Cosmological Perturbations and the Bondi–Sachs Formalism

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Outline

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Motivation

- General Relativity is our best mathematical model for classical gravity.
- With $(\mathcal{M}, g)$ a *spacetime*, g obeys the Einstein equations.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{1}$$

- Einstein equations are *hard!* How to handle?
  - Exact solutions: Black holes, cosmologies, maximally symmetric spacetimes, instantons, ...
  - Series solutions???

Linearized Perturbation Theory

- Consider the ansatz:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + G\delta g_{\mu\nu} + \mathcal{O}(G^2)\,, \tag{2}$$

- e.g., $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, $T_{\mu\nu} = \Lambda = 0 \implies$ perturbations around flat vacuum + no matter.

$$-\eta^{\rho\sigma}\partial_\rho\partial_\sigma\delta\hat{g}_{\mu\nu} + \partial_\mu\partial_\rho\delta\hat{g}^\rho_\nu + \partial_\nu\partial_\rho\delta\hat{g}^\rho_\mu - \eta_{\mu\nu}\partial_\rho\partial_\sigma\delta\hat{g}^{\rho\sigma} = 0\,, \tag{3}$$

  with $\delta\hat{g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\eta^{\rho\sigma}\delta g_{\rho\sigma}\eta_{\mu\nu}$.

- Fix gauge conditions $\partial_\mu\delta\hat{g}^{\mu\nu} = \delta\hat{g}_{00} = \delta\hat{g}_{i0} = 0$ to get

$$\partial_\rho\partial^\rho\delta g_{\mu\nu} = \Box_\eta\delta g_{\mu\nu} = 0\,. \tag{4}$$

How Well do Linearized Methods Fare?

- Pros
  1. Practical + Robust.
  2. Useful for symmetry and stability analyses.

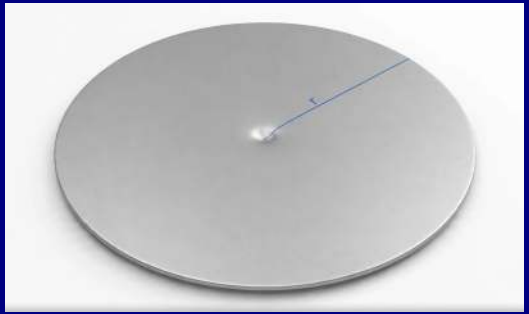
- Cons
  1. Existence, convergence and non–linear uplifts?
  2. Gauge–invariance and the meaning of gravitational energy?

Solution? $\Longrightarrow$ *Expand around infinity!*

Penrose Completions– Intuition



$\eta_{\mu\nu}$



$\eta_{\mu\nu} + o(1)$

Penrose Completions– Formalities

- **Defn:** We say that a spacetime $(\mathcal{M}, g)$ possesses a conformal boundary if it smoothly embeds (via F) into a spacetime with boundary $(\tilde{\mathcal{M}}, \tilde{g})$ such that
  1. $\mathcal{M} \cong \mathrm{Int}(\tilde{\mathcal{M}})$ and $F_* \tilde{g} = \Omega^2 g$ for some $\Omega \in C^\infty(\tilde{\mathcal{M}})$,
  2. $\Omega = 0$ on $\partial\tilde{\mathcal{M}}$, $d\Omega \neq 0$ on $\partial\tilde{\mathcal{M}}$.

- If $||d\Omega||_{\tilde{g}} < 0$ on $\partial\tilde{\mathcal{M}}$, then the boundary is *spacelike* e.g. de Sitter spacetime.

- If $||d\Omega||_{\tilde{g}} > 0$ on $\partial\tilde{\mathcal{M}}$, then the boundary is *timelike* e.g. Anti–de Sitter spacetime.

- If $||d\Omega||_{\tilde{g}} = 0$ on $\partial\tilde{\mathcal{M}}$, then the boundary is *null* e.g. flat spacetime.

Asymptotic Expansions in General Relativity–General Philosophy

- Sequence of steps as follows
  1. Find Penrose completion and demarkate boundary as spacelike/timelike/null.
  2. Work out general series expansion of metric in Ω or in $r \equiv \Omega^{-1}$.
  3. Solve Einstein equations order–by–order in radial expansion parameter.
- 'Perturbative' in Ω or $1/r$, non–perturbative in G.

Asymptotic Expansions in General Relativity–The Anti–de Sitter Example

- AdS metric given by (with $x^M = \{\Omega, x^\mu\}$)

$$ds^2 = g_{MN} dx^M dx^N = \frac{d\Omega^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{\Omega^2}\,. \tag{5}$$

- Consider general even, $d+1$–dimensional *Einstein* spacetimes admitting a conformal completion with timelike boundary.
- Metric given by [Fefferman–Graham]

$$ds^2 = g_{MN} dx^M dx^N = \frac{d\Omega^2 + g_{\mu\nu}(\Omega, x) dx^\mu dx^\nu}{\Omega^2}\,, \tag{6}$$

- with

$$g_{\mu\nu}(\Omega, x) = g^0_{\mu\nu}(x) + g^2_{\mu\nu}(x)\Omega^2 + ... + g^{d-1}_{\mu\nu}(x)\Omega^{d-1} + g^d_{\mu\nu}(x)\Omega^d + ...\,, \tag{7}$$

- $g^i_{\mu\nu}$, $i \in \{1, ...d-1\}$ related to curvatures of $g^0_{\mu\nu}$. $g^d_{\mu\nu}$ ambiguous, *but* $\nabla^\mu_{g^0} g^d_{\mu\nu} = 0$!

Asymptotic Expansions in General Relativity–The Bondi–Sachs Formalism

- Aim is to play the same game for flat spacetimes. in 3+1D, Penrose boundary is $\mathbb{R} \times S^2$.
- $\mathbb{R}$ is 'null time' u. S^2 is the 'celestial sphere'.
- Flat spacetime is

$$ds^2 = -du^2 + 2\frac{dud\Omega}{\Omega^2} + \frac{q_{AB}dx^A dx^B}{\Omega^2} = -du^2 - 2dudr + r^2 q_{AB}dx^A dx^B . \qquad (8)$$

- *Asymptotically* flat spacetimes? [Bondi, Van der Burg, Metzner], [Sachs]

$$ds^2 = -\left(1 - \frac{2m_b}{r} + \mathcal{O}(r^{-2})\right)du^2 - 2(1 + \mathcal{O}(r^{-2}))dudr \qquad (9)$$
$$+ \left(r^2 q_{AB} + rC_{AB} + \mathcal{O}(1)\right)dx^A dx^B + \left(\frac{4N_A + 3u\partial_A m_b}{3r} + \mathcal{O}(r^{-2})\right)dudx^A .$$

- How to interpret the expansions? Solving Einstein equations gives

$$\partial_u m_b = \frac{1}{4}\nabla^A\nabla^B N_{AB} - \frac{1}{8}N^{AB}N_{AB}, \tag{10a}$$

$$\partial_u N_A = -\frac{1}{4}\nabla^B\left(\nabla_B\nabla^C C_{AC} - \nabla_A\nabla^C C_{BC}\right) - u\partial_A\partial_u m_b + \frac{1}{4}\partial_A\left(C_{BC}N^{BC}\right) - \frac{1}{4}\nabla_B\left(C^{BC}N_{CA}\right) + \frac{1}{2}C_{AB}\nabla_C N^{BC}. \tag{10b}$$

- Physics? *Mass and angular momentum are carried away by gravitational radiation!*

Cosmological Spacetimes

- Consider instead spacetimes of the form

$$ds^2 = (u+r)^{\frac{2s}{1-s}} \left(-du^2 - 2dudr + r^2 q_{AB} dx^A dx^B \right) , \tag{11}$$

  with $s \in [0,1)$. These are *spatially flat decelerating Friedmann–Robertson–Walker*.

- $s = 1/2 \implies$ radiation dominated Universe.

- $s = 2/3 \implies$ matter dominated Universe.

- Such spacetimes admit null boundaries. [Bonga, Prabhu]

- Can we import tricks from the flat space Bondi–Sachs formalism?

Cosmological Bondi–Sachs...or maybe not?

- Below ansatz for asymptotically *FRW* spacetimes?

$$ds^2 - (u+r)^{\frac{2s}{1-s}}\left[-\left(1-\frac{2m_b}{r}+\mathcal{O}(r^{-2})\right)du^2 - 2(1+\mathcal{O}(r^{-2}))dudr \right. \tag{12}$$
$$\left. + \left(r^2 q_{AB} + rC_{AB} + \mathcal{O}(1)\right)dx^A dx^B + \left(\frac{4N_A + 3u\partial_A m_b}{3r} + \mathcal{O}(r^{-2})\right)dudx^A\right].$$

- Issue? m_b and N_A *not* mass + angular momentum! Fixed via C_{AB} + boundary geometry.
- Solution? Novel *double* expansion in $1/r$ and $1/a = (u+r)^{\frac{-2s}{1-s}}$. [Campaleoni, Delfante, Geiller, Maindiaux]
- Nature of expansions poorly understood. Exemplar solutions lacking! Where to look?

Cosmological Perturbations to Cosmological Bondi–Sachs

- Consider $g_{\mu\nu} = \bar{g}^{\mathrm{FRW}}_{\mu\nu} + G\delta g_{\mu\nu} + \mathcal{O}(G^2)$.
- Linearized equations decouple into scalar, vector, tensor modes.
- Strategy: [Nenmeli, Vaibhav (in preparation)]
  1. Solve for SVT modes in spherical polar coordinates.
  2. Map polar radial coordinate to Bondi–Sachs radial coordinate.
  3. Rewrite metric up to $\mathcal{O}(G)$ in Bondi–Sachs form.
  4. Clarify nature of radial expansions.

- Results:
  1. Resulting expressions very subtle:
     1. Scalar mode sources double radial expansions in $1/r$, $1/a^{\frac{1-s}{s}}$.
     2. Vector mode sources radial expansions in just $1/r$ (bar an overall $1/a^2$ factor).
     3. Tensor mode sources double radial expansions in $1/r$, $1/a^{\frac{-s}{1+s}}$.
  2. Can read off mass/energy and angular momentum for our solutions.
  3. Vector modes contribute to mass, angular momentum, defying naïve CPT expectations.
  4. Can regenerate expected asymptotic symmetry algebra.

Conclusions

1. Series expansions powerful tools for understanding solution spaces of Einstein equations.
2. G expansions, asymptotic expansions provide complementary POVs on relevant physics.
3. Bondi–Sachs well understood for flat asymptotics, not in cosmological settings.
4. We can use G expansions to probe non–trivial aspects of cosmological Bondi–Sachs.
5. Future work?
    1. Soft theorems.
    2. Memory Effects.
    3. Infrared triangles and much more...,

Thanks for your time!