Shape modes of critically coupled vortices

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The nonlinear gauged sigma model

- Oriented Riemannian surface Σ ($\mathbb{R}^2$)
- Kähler manifold (target) (M, J, g, ω)
- Hamiltonian action ρ of $U(1)$ on M
- Connection A on a principal $U(1)$-bundle $P \to \Sigma$ (real-valued 1-form)
- Section ϕ (Higgs field) of the associated bundle $P \times_\rho M \to \Sigma$ ($\phi : \Sigma \to M$)

We are interested in finding pairs (A, ϕ) which are global minimizers of the Yang-Mills-Higgs energy functional

$$E[\phi, A] = \frac{1}{2} \int_\Sigma \left[|F|^2 + |D\phi|^2 + |\mu \circ \phi|^2 \right], \tag{1}$$

where $F = dA$ (curvature of connection), $D\phi = d\phi - A \otimes X \circ \phi$ (covariant derivative), $\mu : M \to \mathfrak{g}^*$ is the moment map, X denotes the generating vector field associated with the $U(1)$ action.

The nonlinear gauged sigma model

- Compatibility: The action is compatible with the complex and Riemannian structures of the target:

$$\mathcal{L}_X g = 0, \quad \mathcal{L}_X J = 0.$$

- Boundary conditions: For the non-compact case $\Sigma = \mathbb{R}^2$, finite energy solutions require B.C.:

$$\lim_{|x|\to\infty} F = 0, \quad \lim_{|x|\to\infty} D\phi = 0, \quad \lim_{|x|\to\infty} \mu \circ \phi = 0.$$

- Gauge transformations: The energy functional (1) is invariant under the local gauge transformations

$$\phi \mapsto g \cdot \phi, \quad A \mapsto A - ig^{-1}dg,$$

where $g : \Sigma \to U(1)$.

- Vortex positions: Points on Σ that are mapped by the Higgs field ϕ to a fixed point of the $U(1)$ action on the target M.

Examples

- Abelian-Higgs model
  - $M = \mathbb{C}$, $U(1)$ acts by rotations about the origin
  - $X = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$, $D\phi = d\phi - iA\phi$
  - J acts by multiplication by i
  - Vortex positions: $\phi(p_i) = 0$
- Gauged $\mathbb{C}P^1$ model
  - $M = \mathbb{C}P^1 \cong S^2$, $U(1)$ acts by rotations about the z-axis
  - $X = \dfrac{\partial}{\partial \varphi}$, $D\boldsymbol{\phi} = d\boldsymbol{\phi} - A\boldsymbol{n} \times \boldsymbol{\phi}$, where $\boldsymbol{n} = (0, 0, 1)$
  - J acts by multiplication by $\boldsymbol{\phi}\times$
  - Vortex positions: $\phi(p_i) = \pm\boldsymbol{n} \rightsquigarrow$ two vortex species!

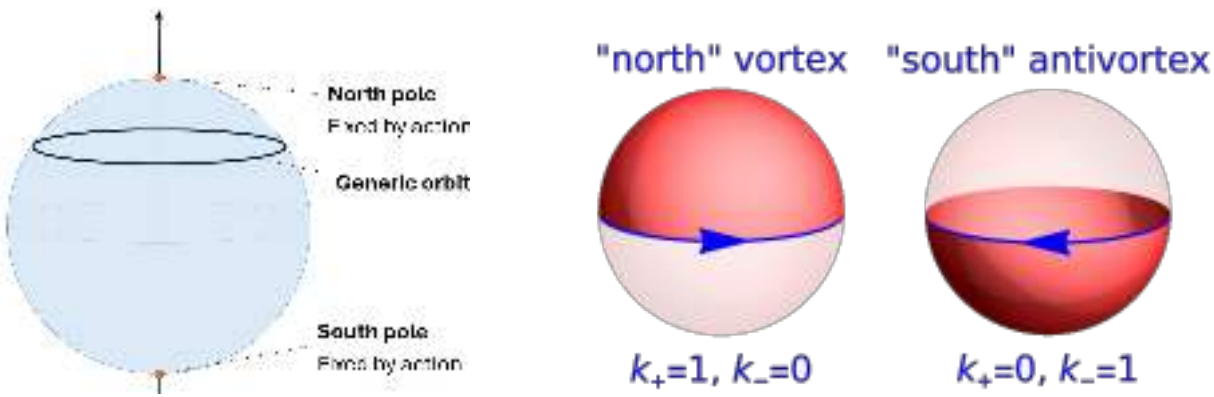


Figure: $\mathbb{C}P^1$ vortex species[1].

[1] J.M. Speight and T. Winyard, Phys. Rev. B 103 (2021), 014514.

The Bogomolny decomposition

An important feature of this model is that it admits a Bogomolny argument. We define the operator

$$Bog : C^\infty(\Sigma, M) \oplus \Gamma(T^*\Sigma) \to \Gamma(T^*\Sigma \otimes \phi^* TM) \oplus C^\infty(\Sigma)$$

$$\begin{pmatrix} \phi \\ A \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}}(D\phi - J \circ \star D\phi) \\ \\ \star F - \mu(\phi) \end{pmatrix}.$$

Then we can re-write the energy more compactly as

$$E[\phi, A] = \frac{1}{2}||Bog(\phi, A)||^2_{L^2} + E_0, \tag{2}$$

where $E_0 = \int_\Sigma \phi^\star \omega + \int_\Sigma d(A\mu(\phi))$, representing a topological invariant[2].

[2] K. Cieliebak, Gaio. A.R., and D.A. Salamon, J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions, arXiv:2603.21343 (1999).

The Bogomolny equations

- From (2) we deduce the energy is minimised when the Bogomolny equations hold

$$D\phi - J \circ \star D\phi = 0, \quad \star F = \mu(\phi), \tag{3}$$

- Solutions of (3) are called BPS solutions.
- In particular

$$D\phi - i \star D\phi = 0, \quad \star F = \frac{1}{2}(1 - |\phi|^2) \quad \text{(Abelian Higgs)}$$

$$D\boldsymbol{\phi} - \boldsymbol{\phi} \times \star D\boldsymbol{\phi} = 0, \quad \star F = -\tau + \boldsymbol{n} \cdot \boldsymbol{\phi} \quad \text{(Gauged } \mathbb{C}P^1\text{)}$$

Theorem (N.S. Manton, M. Speight, 2006)

If M is compact, then there exists a solution to the Bogomolny equations on $\Sigma = \mathbb{R}^2$.

Shape modes: a geometric formalism

- Shape modes = eigenmodes of the Jacobi operator with positive eigenvalue
- Jacobi operator $\mathscr{J} = \mathscr{B}^\dagger \mathscr{B}$, where $\mathscr{B}$ is the differential of the map *Bog* at (ϕ, A):

$$\mathscr{B} : \mathcal{PERT} \to \Gamma(T^*\Sigma \otimes \phi^* TM) \oplus C^\infty(\Sigma)$$

$$\begin{pmatrix} \eta \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} L_J\left(\nabla^\phi \eta - \alpha \otimes (X \circ \phi) - A \otimes (\nabla_\eta X) \circ \phi\right) \\ \star d\alpha - d\mu|_\phi(\eta) \end{pmatrix}$$

and $\mathscr{B}^\dagger$ is its L^2-adjoint.

- Jacobi operator = product of two first order operators, simply due to the fact that the model admits a Bogomolny decomposition.

Some results

If $\psi : \Sigma \to \mathbb{R}$ is an L^2-eigenfunction of the Schrödinger operator

$$L = \Delta + ||X \circ \phi||_g^2 \tag{4}$$

then $\mathcal{S}_1 \mathscr{G} \psi$ is an L^2-integrable eigenfunction of $\mathscr{J}$ with the same eigenvalue. Here $\Delta = - \star d \star d$ denotes the Laplace-Beltrami operator on Σ.

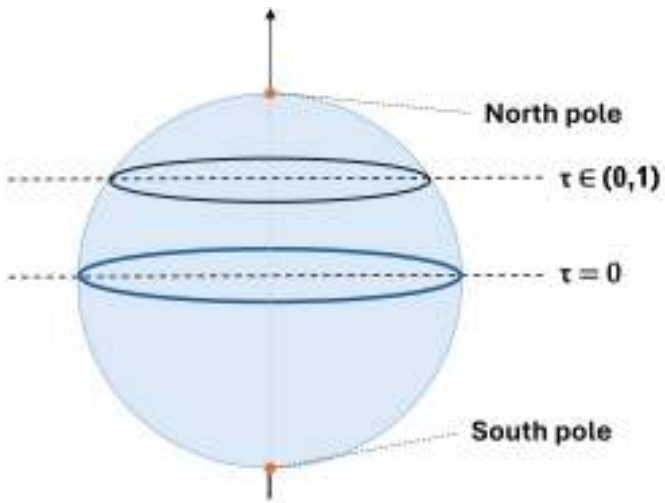
- Powerful and quick method to construct a shape mode by simply solving a single scalar PDE on Σ
- But...do *all* shape modes arise in this way?

If $M \cong S^2$, then there exists a bijective correspondence between the eigenfunctions of $\mathscr{J}$ and L.

Some results

Let M be compact. Then there exists a choice of normalization of the moment map μ such that every BPS vortex has a shape mode on $\Sigma = \mathbb{R}^2$.

Example: The gauged $\mathbb{C}P^1$ model.
Moment map $\mu(\boldsymbol{\phi}) = -\tau + \boldsymbol{n} \cdot \boldsymbol{\phi}$,
$\tau \in (-1, 1)$.
Choosing $\tau = 0$, one can show[3] that any $\mathbb{C}P^1$ vortex has a shape mode.



[3] N. Gavrea, D. Harland, and M. Speight, Shape modes of $\mathbb{C}P^1$ vortices, Lett Math Phys 116, 126 (2026).

Thank you!