Bow Varieties and the Geometry of Bad QFTs

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Based on work with M. Fazzi,
to appear arXiv:2611.xxxx

AGQ Student Conference 2026



Why study 3d $\mathcal{N} = 4$ Theories?



Physics

- Strongly coupled IR physics via dualities:

  $$\text{strong} \longleftrightarrow \text{weak}$$

- High SUSY $\implies$ robust analytic tools
- Controlled breaking
  $\mathcal{N} = 4 \to 2 \to 1 \to 0$
  e.g. particle-vortex [Kachru et al. '16]

Mathematics

- "Physical Mathematics":
  Physics intuition $\implies$ non-trivial maths dualities
- 3d $\mathcal{N} = 4$ Mirror Symmetry
  [Intriligator-Seiberg '96]
  $\implies$ Symplectic duality [BLPW '14]
  $\implies$ Koszul duality [BGS '96]

3d $\mathcal{N} = 4$ Lightning Review

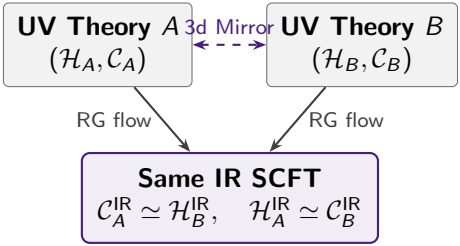
3d $\mathcal{N} = 4$ basics:

- 8 supercharges
- R-symmetry: $\mathfrak{so}(6) \simeq \mathfrak{su}(2)_H \times \mathfrak{su}(2)_C$

Multiplet:	**Hyper**	**Vector**
Scalars	$(q, \tilde{q}^\dagger)$	$\vec{\Phi}$, dual photon σ
$SU(2)_H \times SU(2)_C$	$(\mathbf{2}, \mathbf{1})$	$(\mathbf{1}, \mathbf{3})$
Branch:	Higgs $\mathcal{H}$	Coulomb $\mathcal{C}$
Geometry	Hyperkähler	Hyperkähler
Nature	**Classically exact**	**Quantum corrected**

(Vacuum moduli parametrised by scalar VEVs!)

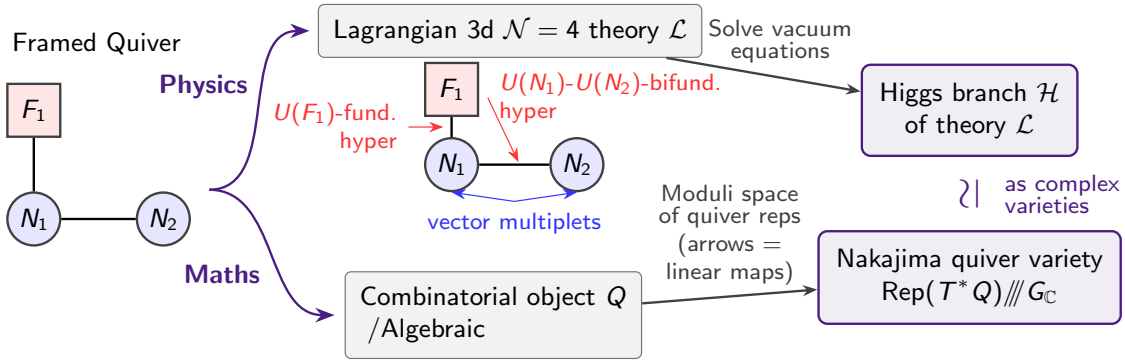
3d Mirror Symmetry



- Computes quantum $\mathcal{C}_A$ as a classical $\mathcal{H}_B$! [Intriligator-Seiberg '96]

3d $\mathcal{N} = 4$ Quiver Theories Review

Quiver Theories and their Higgs branches:

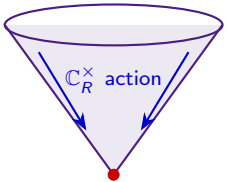


Coulomb branch of Quiver Theories:

Direct computations is quantum and difficult; approaches in [CHZ 13', BDG '15, BFN '16]

Geometry of 3d $\mathcal{N} = 4$ and SCFTs

3d $\mathcal{N} = 4$ Physics	**Moduli Space Geometry $\mathcal{M}$**
Hyperkähler space of vacua	**Symplectic singularity** [Beauville '00]
$U(1)_R \subset SU(2)_R$ symmetry	$\mathbb{C}^\times$-action on $\mathcal{M}$
BPS operators $\mathcal{O}$	Coordinate ring $\mathbb{C}[\mathcal{M}] = \bigoplus_d \mathbb{C}[\mathcal{M}]_d$
SCFT unitarity ($\Delta = R \geq \frac{1}{2}$)	Positive grading ($d \geq 0$, $\mathbb{C}[\mathcal{M}]_0 = \mathbb{C}$)
Unique conformal vacuum	**Conical symplectic singularity**

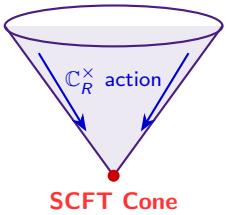


What if $\mathcal{M}_{\text{UV}}$ is NOT a cone?

How to extract the IR SCFT?

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How to extract the IR SCFT?

Geometric RG-flow:

UV Moduli $\mathcal{M}_{\mathbf{UV}}$
(Not a cone in general)

It is the IR NLSM target

Pick vacuum $x \in \mathcal{M}_{UV}$

Zoom in on $x \in \mathcal{M}_{\mathbf{UV}}$
by local completion.
Induces decomposition
+ canonical good $\mathbb{C}^\times_R$-action

$\mathbb{C}^\times \curvearrowright \widehat{\mathcal{M}_x} \simeq \widehat{\mathcal{L}_x} \times \widehat{\mathcal{S}_x}$

Extract CFT

Chiral Ring of deep IR CFT:
Take weighted tangent cone
$\mathcal{M}_x^{\mathsf{IR}} := \mathrm{Spec}(\mathrm{gr}_{\mathbb{C}^\times} \widehat{\mathcal{M}_x})$
w/ $\mathbb{C}^\times$-weights $\Rightarrow$ IR R-charges

Geometric RG-flow and Quiver Theories

Gaiotto-Witten Classification of quivers [Gaiotto-Witten '08]: ($e_i := F_i + \sum_{j\sim i} N_j - 2N_i$)

Property	**Good Quiver Theory**	**Bad Quiver Theory**
Balance Condition	Satisfies $e_i \geq 0 \ \ \forall i$	Contains nodes where $e_i < 0$
Monopole Operators	$\Delta \geq \frac{1}{2}$ (satisfies unitarity bound)	$\Delta < \frac{1}{2}$ (violates apparent unitarity bound)
Higgs Branch $\mathcal{H}$	Quiver variety $\Rightarrow$ Cone	Quiver variety $\Rightarrow$ Cone
Coulomb Branch $\mathcal{C}$	Cone $\simeq \mathcal{H}_{\text{mirror}}$ (Quiver variety)	**NOT a cone $\neq$ Quiver variety! What is Coulomb branch geometry?**
IR SCFT Vacuum	Canonical apex of $\mathcal{C}$ and $\mathcal{H}$	Smooth $\mathcal{M}$ / many intersecting cones / extended singularity
3d Mirror B	Standard Quiver Theory $\Longrightarrow \mathcal{C}_A \simeq \mathcal{H}_B$ (quiver variety)	

Geometric RG-flow and Quiver Theories

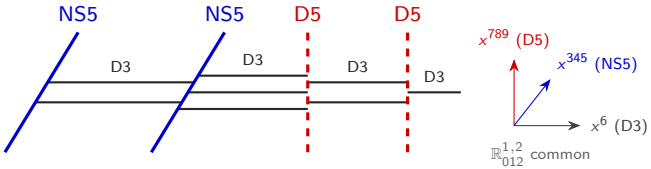
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3d Mirror B	Standard Quiver Theory $\Longrightarrow \mathcal{C}_A \simeq \mathcal{H}_B$ (quiver variety)	**Bow Theory!**

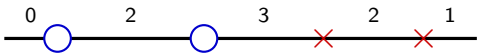
Bow Diagrams from Brane Engineering

Hanany-Witten Setup in Type IIB [HW '96]

Brane Setup:



Bow Diagram:

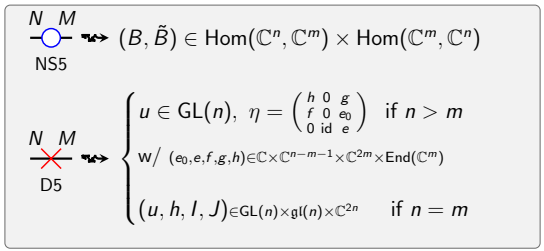


- The worldvolume of N D3s hosts the EFT 4d $\mathcal{N} = 4$ $U(N)$ SYM
- Intersections with NS5s and D5s host gauge (bi)fundamental matter breaking 4d $\mathcal{N} = 4 \to 2$ [Cherkis et al. '11]

Higgs Branch of Bow Theories

[Cherkis '10, Nakajima-Takayama '16]

- Quivers generalisation:
  Solving 4d QFT $\iff$ Moduli of reps of vacuum eqns bow diagrams
- Bow variety of bow diagram $\mathcal{B}$:
  $\mathrm{Rep}(\mathcal{B}) /\!\!/\!\!/ G_{\mathbb{C}}$



3d Bow QFT: KK reduction of the 4d $\mathcal{N} = 2$ EFT along x^6 yields a 3d $\mathcal{N} = 4$ QFT

- D3 $\implies$ 3d $\mathcal{N} = 4$ gauge vector multiplets
- NS5 $\implies$ 3d $\mathcal{N} = 4$ $U(N_L)$-$U(N_R)$-bifundamental hypermultiplet
- Cobalanced ($N_L = N_R$) D5 $\implies$ 3d $\mathcal{N} = 4$ $U(N_L)$-fundamental hypermultiplet
- Non-cobalanced ($N_L \neq N_R$) nodes have no 3d UV Lagrangian terms!

Bow Variety of cobalanced bow $\simeq$ Quiver Variety

3d Mirror Symmetry

Type IIB S-duality exchanges NS5 ($\bigcirc$) $\longleftrightarrow$ D5 ($\times$)

Bow Diagram $\mathcal{B}$ **Mirror Bow Diagram $\mathcal{B}^\vee$**



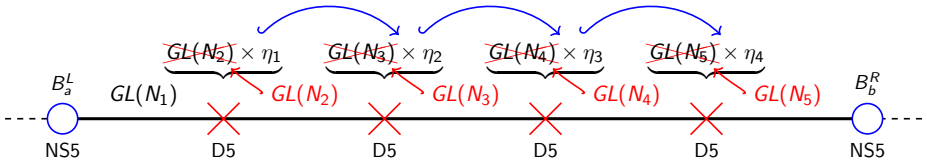
- Bow theories are closed under 3d Mirror Symmetry!
- Exact quantum Coulomb branch of any bow can be computed classically as the mirror bow:

$$\mathcal{C}(B) \simeq \mathcal{H}(\mathcal{B}^\vee)$$

- Bow Theories of cobalanced bow diagrams $\iff$ 3d Quiver Theories! (good or bad)

Lagrangians for (Monotonic) Bow Theories

Assume $N_1 < N_2 < N_3 < N_4 < N5$ betweem two NS5s $\implies$ 'bow segment' telescopes to a constraint on B^R: $B^R \tilde{B}^R = \eta_4$



- **Monotonic bow:** each interval between NS5s has monotonic N_i's
- Monotonic bows have $3d\mathcal{N} = 2$ Lagrangians via flipping fields λ and superpotential terms

$$\Delta W = \mathrm{Tr}(\sum_{ij} \lambda_{ij}(B\tilde{B}_{ij} - \eta_{ij})) \qquad (i,j \text{ summing entries} = 0 \text{ or } 1)$$

- In IR SUSY enhances back to $\mathcal{N} = 4$!

Examples: Coulomb Branches of Bad Quivers as Mirror Bows

Example 1: $T[U(N)]$-$T[U(N)]$

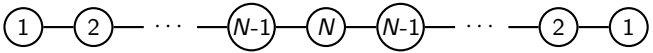
'Higgs branch' gluing of two $T[U(N)]$

Coulomb branch:

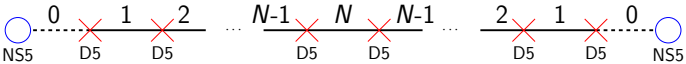
$$\mathrm{T}^*\mathrm{GL}(N) \simeq \mathrm{GL}(N) \times \mathfrak{gl}(N)$$

$\Rightarrow$ in deep IR non-interacting at any vacuum

Quiver of the $T[U(N)]$-$T[U(N)]$ theory:



Mirror bow: (of the bow corresponding to the initial quiver)

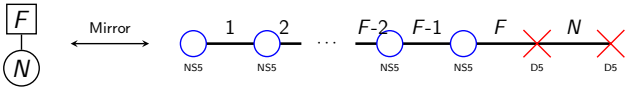


Example 2:
$U(N)$-SQCD with $U(F)$-flavour

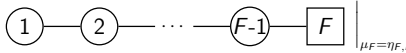
Coulomb branch:
$T[U(F)]$ quiver variety deformed by $U(F)$-flavour moment map to

- another cone for $F > 2N$ (Good)
- curved space for $F < 2N$ (Bad)

Quiver of SQCD and its mirror bow:



SQCD Coulomb branch in terms of quiver geometry:



The Universe of Computing Bad 3d $\mathcal{N} = 4$ Coulomb Branches

Partition Function Identities
[Pasquetti Group '21]

S_b^3 / index identities decompose bad theory into a **sum over good theories**: $Z_{\text{bad}} = \sum_r \delta_r \, Z_{\text{good}}^{(r)} \cdot Z_{\text{free}}^{(r)}$

Magnetic Quivers & Hilbert Series
[Grimminger-Hanany '20]

Local cones approximated by a **collection of magnetic quivers** & symplectic leaves.

IR Abelianisation / BFN
[BDG '15, BFN '16, Assel-Cremonesi '17]

Direct construction of quantum Coulomb branch chiral ring from the **abelianisation formula**.

Bow Varieties

- Explicitly global geometry $\Rightarrow$ Stratification follows
- Only approach with a classical computation: as Higgs branch of the mirror
- Significantly simpler than BFN, more rigorous than BDG
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