Chapter 6 - Stabilisation or absorption

R. Roenig

Question: Can we classify all Hilb. A-mod. up to unitary equivalence (iso.)?

↳ we need to restrict the size of A and E.

Notation: Unless otherwise stated: A, B C^*-alg., E Hilb. A-mod. $\mathcal{H}$ separable Hilb sp. ($\dim \mathcal{H} = \infty$).

Def. $h \in A_+$ is strictly positive ($h > 0$) if $\rho(h) > 0$ $\forall \rho \in S(A)$.

Lemma 6.1. $h > 0 \iff \overline{hA} = A$ (closed right ideal gen by h).

Pf. ($\Leftarrow$) (contradiction) Suppose $\exists \rho \in S(A)$ s.t. $\rho(h) = 0$ and $\overline{hA} = A$.

Then, $|\rho(ha)|^2 = |\rho(h^{\frac{1}{2}} h^{\frac{1}{2}} a)|^2 \overset{CS}{\leq} \rho(h)\,\rho(a^* h a)$ $\forall a \in A$

$\Rightarrow \rho|_{\overline{hA}} \equiv 0$ ↯ contradiction! ("big theorem")

($\Rightarrow$) (contrapositive) Suppose $\overline{hA} \neq A \Rightarrow \exists \rho \in S(A)$ st. $\rho|_{\overline{hA}} \equiv 0 \Rightarrow \rho(h) = 0$. ∎

Def. A is σ-unital if $\exists$ ctbl. app. unity (i.e., $(e_n) \subseteq A$...)

Rmk: A σ-unital $\iff \exists h > 0$. ($\Rightarrow$ take $h = \sum_n 2^{-n} e_n$; $\Leftarrow$ if $h > 0$, $(h^{\frac{1}{n}})_n$ is an app. unit.)

Ex: (i) $C_0(X)$ σ-unital $\iff$ X is σ-cpt. (X 2nd ctbl)

(ii) A separable $\Rightarrow$ A σ-unital

Def $Z \subseteq E$ is a generating set if the closed submodule generated by Z is equal to E.

Ex (i) A σ-unital $\Leftrightarrow$ A is ctbly gen. (as an A-mod.)

($\Rightarrow$) Given $h>0$, we have $\overline{hA}=A$ by Lemma 6.1. Hence $\{h\}$ is a gen. set.

($\Leftarrow$) If $\{a_n\}_n$ is a gen. set (WLOG $a_n \geq 0$ & $\|a_n\| \leq 1$), we have $h := \sum_n 2^{-n} a_n > 0$ (indeed, $\rho(h)=0 \Rightarrow \rho(a_n)=0 \ \forall n \Rightarrow \rho \equiv 0$, $\rho \in S(A)$).

(ii) A σ-unital $\Rightarrow \mathcal{H}_A := \mathcal{H} \otimes A$ is ctbly gen.

Consider $\{\xi_n\}$ basis of $\mathcal{H}$, $h \in A$, $h>0 \rightsquigarrow \{\xi_n \otimes h\}$ gen set of $\mathcal{H}_A$.

Recall Prop 3.8 $t \in \mathcal{L}(E,F)$ s.t. $\overline{\mathrm{ran}(t)}=F$, $\overline{\mathrm{ran}(t^*)}=E \Rightarrow E \cong F$.

Thm 6.2 [Kasparov's absorption or stabilisation] ("$\mathcal{H}_A$ is big enough to absorb")

E ctbly gen A-mod $\Rightarrow E \oplus \mathcal{H}_A \cong \mathcal{H}_A$. ("$E$ cannot be bigger than $\mathcal{H}_A$")

Pf: We can reduced it to the case where A is unital.

- $\{y_n\}^{\subseteq E}$ seq. of unit vectors in which each element of a gen. set occurs infinitely often
- $\{e_n = \xi_n \otimes 1\}$ gen. set of $\mathcal{H}_A$.

Define $t \in K(\mathcal{H}_A, E \oplus \mathcal{H}_A)$ by $t := \sum_k \left(2^{-k}\theta_{y_k,e_k} + 4^{-k}\theta_{e_k,e_k}\right)$ ($\theta_{x,y}(z) = x\langle y,z\rangle$)

$\therefore te_n = 2^{-n}\underbrace{y_n}_{\in E} + 4^{-n}\underbrace{e_n}_{\in \mathcal{H}_A}$

For each m with $y_m = y_n$ we have

$$t(2^m e_m) = y_m + 2^{-m} e_m = y_n + 2^{-m} e_m \xrightarrow[\text{"}m \to \infty\text{"}]{\text{infinitely many } m} y_n \in \overline{\operatorname{ran}(t)}.$$

Also, $e_n = t(4^n e_n) - 2^n y_n \in \overline{\operatorname{ran}(t)}$.

Thus, $\overline{\operatorname{ran}(t)} = E \oplus \mathcal{H}_A$.

$e_n \in \overline{\operatorname{ran}(t^*)}$

Similarly, $t^* := \sum_k (2^{-k} \Theta_{e_k, y_k} + 4^{-k} \Theta_{e_k, e_k})$ satisfies $t(4^n e_n) = e_n$.

By Prop 3.8, $E \oplus \mathcal{H}_A \cong \mathcal{H}_A$. ∎

Def. $F \leq E$ (closed) is fully complemented if F is complemented in E and $F^\perp \cong E$ (i.e., $F \oplus F \cong E$).

Cor 6.3 E ctbly gen. A-mod $\Rightarrow$ E is unitarily equiv. to a fully complem. submod of $\mathcal{H}_A$.

Rmk: Claim $z = \sum_{i=1}^{k} x_i \otimes y_i \in E \odot F$, $\langle z, z \rangle = 0 \Rightarrow z = 0$.

Idea: Reduce to $z \in E_1 \odot F_1$, where E_1 is gen. by $\{x_i\}$ and F_1 gen. by $\{y_i\}$.

Cor 6.3 ⇝ WLOG $z \in \mathcal{H}_A \odot \mathcal{H}_B$ (because $E_1 \hookrightarrow \mathcal{H}_A$ & $F_1 \hookrightarrow \mathcal{H}_B$).

Using $\mathcal{H}_A \cong \bigoplus_{n \geq 1} A$, $x_i = (a_{in})$; $\mathcal{H}_B \cong \bigoplus_{n \geq 1} B$, $y_i = (b_{in})$

$$0 = \langle z, z \rangle = \sum_{i,j,n} a_{in}^* a_{jn} \otimes b_{in}^* b_{jn} = \sum_n \Big(\sum_i a_{in} \otimes b_{in} \Big)^* \Big(\sum_j a_{jn} \otimes b_{jn} \Big)$$

$\Rightarrow \sum_i a_{in} \otimes b_{in} = 0 \ \forall n \Rightarrow z = 0$.

Rmk: $\mathcal{K}(\mathcal{H}_A) \cong \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(A) \cong \mathbb{K} \otimes A$ stable alg. of A

$\mathcal{L}(\mathcal{H}_A) \cong M(\mathbb{K} \otimes A)$ ↪ important in K-theory.

Lemma 6.4 A sep. C^*-alg, B C^*-alg $\Rightarrow \exists$ faithful non-deg $*$-hom $\pi: A \to \mathcal{L}(\mathcal{H}_B)$.

Pf. A sep $\Rightarrow \exists\, \pi_0: A \to B(\mathcal{H})$ faithful non-deg $*$-rep.

Define $\pi(a) := \pi_0(a) \otimes 1 \in \mathcal{L}(\mathcal{H}) \otimes M(B) \subseteq \mathcal{L}(\mathcal{H} \otimes B)$ ∎

↳ Chapter 4

Thm 6.5 A sep., B σ-unital, $\rho: A \to \mathcal{L}(\mathcal{H}_B)$ non-deg c.p. map.

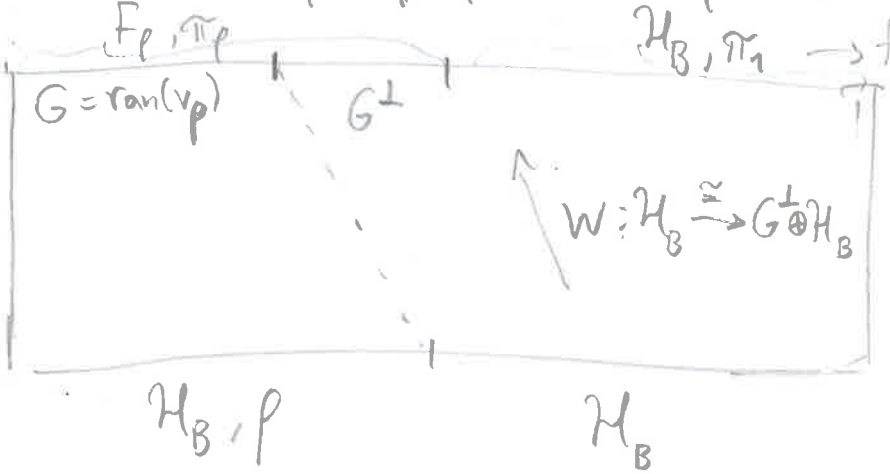
$\Rightarrow \exists$ faithful non-deg. $*$-hom $\pi: A \to M_2(\mathcal{L}(\mathcal{H}_B))$ s.t.

$\rho(a) = \pi(a)_{11}$.

Pf. $\mathcal{H}_B, \rho$ ctbly gen. $\overset{\text{KSGNS}}{\rightsquigarrow} (F_\rho, \pi_\rho, V_\rho)$ $\therefore V_\rho: \mathcal{H}_B \to F_\rho$ isometry

$F_\rho \cong A \otimes_\rho \mathcal{H}_B$

$p: F_\rho \twoheadrightarrow G$ proj.



$\pi(a) := (V_\rho^* \rho^* \oplus W^*)(\pi_\rho(a) \oplus \pi_1(a))(\rho V_\rho \oplus W) \in \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$

$\therefore \pi(a)(X \oplus 0) = V_\rho^* \pi_\rho(a) V_\rho(X) \oplus __ = \rho(a) X \oplus __$ ∎

Rmk: ∃ version of the thm if π is degenerate

Rmk: $M_2(\mathcal{L}(\mathcal{H}_B)) \cong \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B) \cong \mathcal{M}(\mathcal{K}(\mathcal{H}_B \oplus \mathcal{H}_B)) \cong$

$\cong \mathcal{M}(M_2(\mathcal{K}(\mathcal{H}_B)) \cong \mathcal{M}(M_2(\mathcal{K} \otimes B)) \cong \mathcal{M}(M_2 \otimes \mathcal{K} \otimes B) \cong \mathcal{M}(\mathcal{K} \otimes B)$

"new" codomain of $\bar{\pi}$ in the thm.

Prop. 6.7 E ctbly gen. A-mod $\iff$ $\mathcal{K}(E)$ is σ-unital.

Pf: We can reduce the proof to the unital case.

($\Rightarrow$) Claim: $\mathcal{K}(\mathcal{H}_A)$ is σ-unital.

$\mathcal{H}_A$ ctbly gen by $\{e_n = \xi_n \otimes \beta_n\}$. Note that $h = \sum_n 2^{-n} \theta_{e_n, e_n} > 0$ in $\mathcal{K}(\mathcal{H}_A)$ $\therefore$ $\mathcal{K}(\mathcal{H}_A)$ is σ-unital.

E ctbly gen. $\xrightarrow{\text{Cor. 6.3}}$ $E \leq \mathcal{H}_A$ is complemented. (considering $E \oplus H_A \cong \mathcal{H}_A$)

Let $p \in \mathcal{L}(\mathcal{H}_A, E)$ be the projection onto E and $(u_n)_n$ be an app. unit of $\mathcal{K}(\mathcal{H}_A)$. Then $(p u_n p^*)_n$ is an app unit for $\mathcal{K}(E)$.

($\Leftarrow$) Let $(v_n)_n$ be an app unit of $\mathcal{K}(E)$. Then

$\forall n \; \exists x_{n,i}, y_{n,i} \in E \; (1 \leq i \leq m_n)$ s.t. $\| \sum_{i=1}^{m_n} \theta_{x_{i,n}, y_{i,n}} - v_n \| < \frac{1}{n}$.

Hence, given $x \in E$, and $\varepsilon > 0$, for n large enough we have

$\| \sum_{i=1}^{m_n} \theta_{x_{n,i}, y_{n,i}} x - x \| < \varepsilon$ (since $v_n x \longrightarrow x$).

Since $\theta_{x_{n,i}, y_{n,i}} x = x_{n,i} \langle y_{n,i}, x \rangle \rightsquigarrow \{x_{n,i} : 1 \leq i \leq m_n, n \geq 1\}$ gen. E. ∎

Prop 6.8 [non-commutative Tietze theorem] A σ-unital, B C^*-alg.

$\alpha: A \to B$ surjective $*$-hom $\Longrightarrow$ $\overline{\alpha}: M(A) \to M(B)$ surjective

↳ canonical extension (Prop. 2.1)

Rmk: In particular,

$\alpha: A \twoheadrightarrow A/I$ surj $\Rightarrow$ $\overline{\alpha}: M(A) \twoheadrightarrow M(A/I)$ surj

but not a quotient map $M(A)/M(I)$

X σ-cpt. (loc. cpt., Hff $\Rightarrow$ normal)

Ex: $A = C_0(X) \rightsquigarrow M(A) = C_b(X)$

$Y \subseteq X$ closed, $I = C_0(X \setminus Y) \rightsquigarrow A/I \cong C_0(Y)$

Then, Prop 6.8 translates to

$$q: C_0(X) \xrightarrow{\text{surj.}} C_0(Y) \Rightarrow \overline{q}: C_b(X) \xrightarrow{\text{surj.}} C_b(Y)$$

↳ restriction map ⇔ "every $f: Y \to \mathbb{C}$ extends to X".