Einstein 4-Manifolds and Special Geometry

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Introduction

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$$\mathrm{Ric}(g) = \lambda g$$

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Central to intersection of modern Geometry & Physics

- Complex geometry & topology
- Algebraic geometry & twistor theory
- Space-time geometry & quantum gravity

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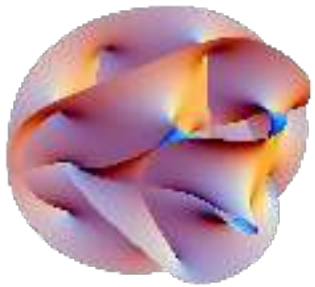
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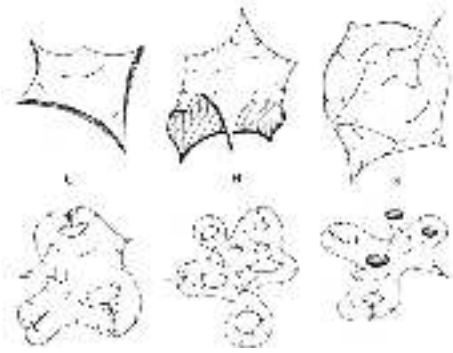
This talk

- Review open problems & major recent progress
- Focus on $\lambda = 0$:

$$\mathrm{Ric}(g) = 0$$

- Do we know any solutions?
- How can we solve the equations?
- Want smooth manifolds, not orbifolds or singular spaces





Special geometry

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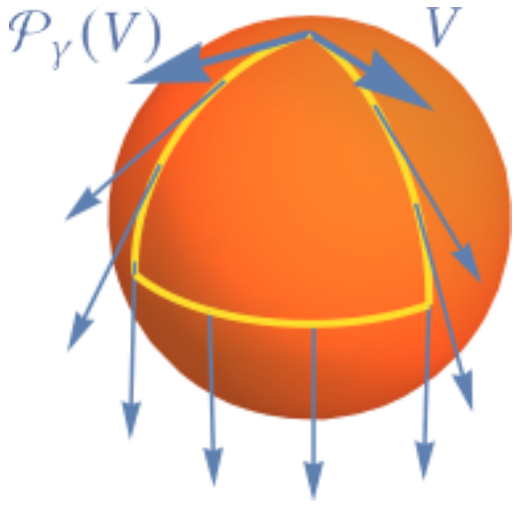
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$$\underbrace{SO(4)}_{\text{generic}}, \qquad \underbrace{U(2)}_{\text{Kähler}}, \qquad \underbrace{SU(2) = Sp(1)}_{\text{Calabi-Yau/hyperkähler}}, \qquad \underbrace{Sp(1) \cdot Sp(1)}_{\text{quaternion-Kähler}}$$

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Closely related to complex geometry:

- Kähler: 1 parallel complex structure
- Hyperkähler: 2-sphere of Kähler structures

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- It is only in 4D that both the Riemann curvature and the Hodge star are maps $\Lambda^2 \to \Lambda^2$

$$\omega_{ab} \mapsto R_{ab}{}^{cd}\omega_{cd}, \qquad \omega_{ab} \mapsto \varepsilon_{ab}{}^{cd}\omega_{cd}$$

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- Riemann tensor $\mathcal{R}$ has irreducible decomposition

$$\mathcal{R} = \begin{pmatrix} W^+ + \frac{R}{12} I & \mathrm{Ric} - \frac{R}{4} g \\ \mathrm{Ric} - \frac{R}{4} g & W^- + \frac{R}{12} I \end{pmatrix}$$

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- A 4-manifold is **half-flat** iff

$$\mathcal{R} = \underbrace{\begin{pmatrix} W^+ & 0 \\ 0 & 0 \end{pmatrix}}_{\text{self-dual}} \quad \text{or} \quad \mathcal{R} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & W^- \end{pmatrix}}_{\text{anti-self-dual}}$$

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What do we actually know?

Need to distinguish two cases:

- Compact
- Non-compact

The compact case

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Example: The 4-torus is

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Open problem: the explicit form of the K3 metric remains unknown!

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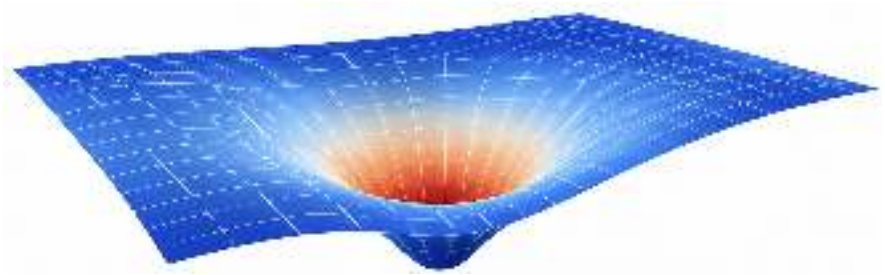
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So the only compact Ricci-flat 4-manifolds we know are half-flat!

The non-compact case

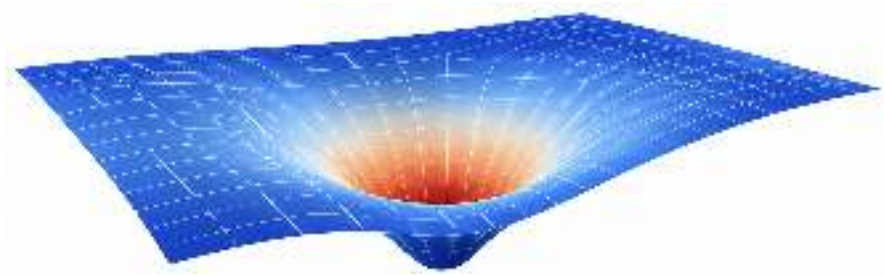
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(M, g) is asymptotic to (M_0, g_0) if $\exists$ compact set K such that

$$M \setminus K \cong M_0, \qquad g \xrightarrow[r\to\infty]{} g_0$$

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- Some examples:

AE / **ALE**	**AF** / **ALF**	**ALG**	**ALH**	(+ others)
r^4	r^3	r^2	r^1	

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$\rightsquigarrow$ completely new, vast and unknown landscape!

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- $\rightsquigarrow$ **Twistor theory?**

Twistor theory

Theorem (Penrose): There is 1-1 correspondence between:

-
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Twistor theory

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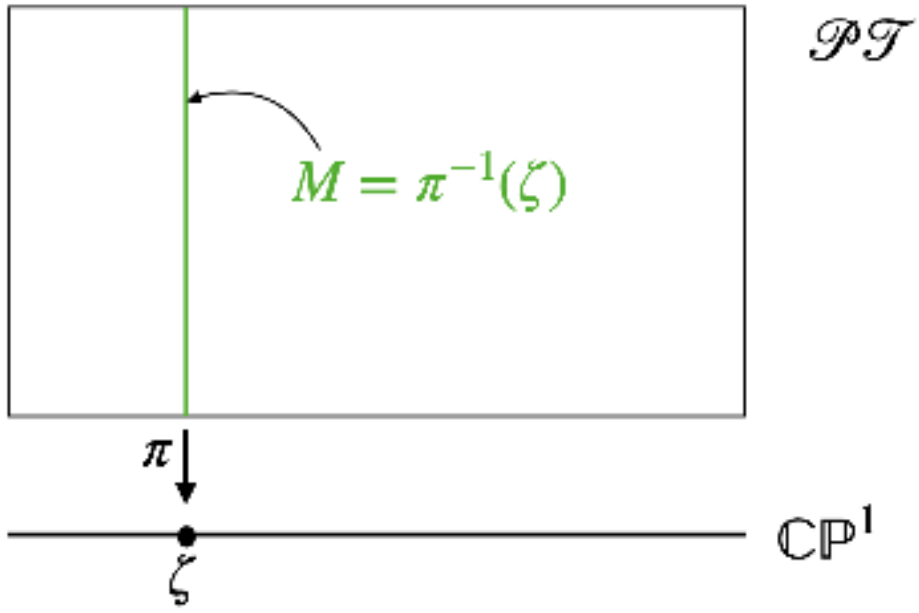
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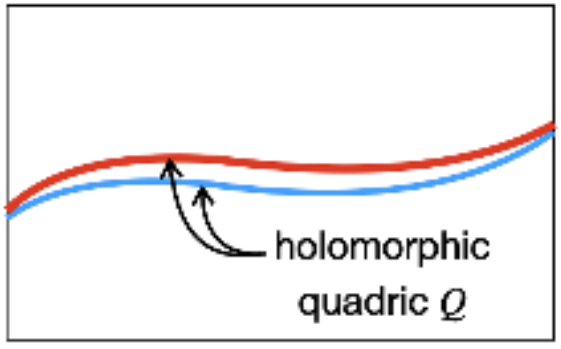
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  Building blocks manifest

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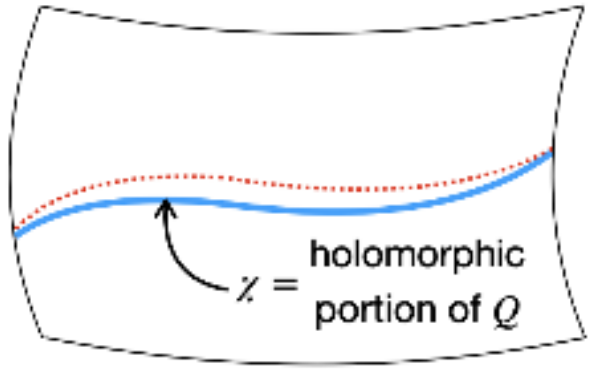


encodes anti-SD Taub-NUT

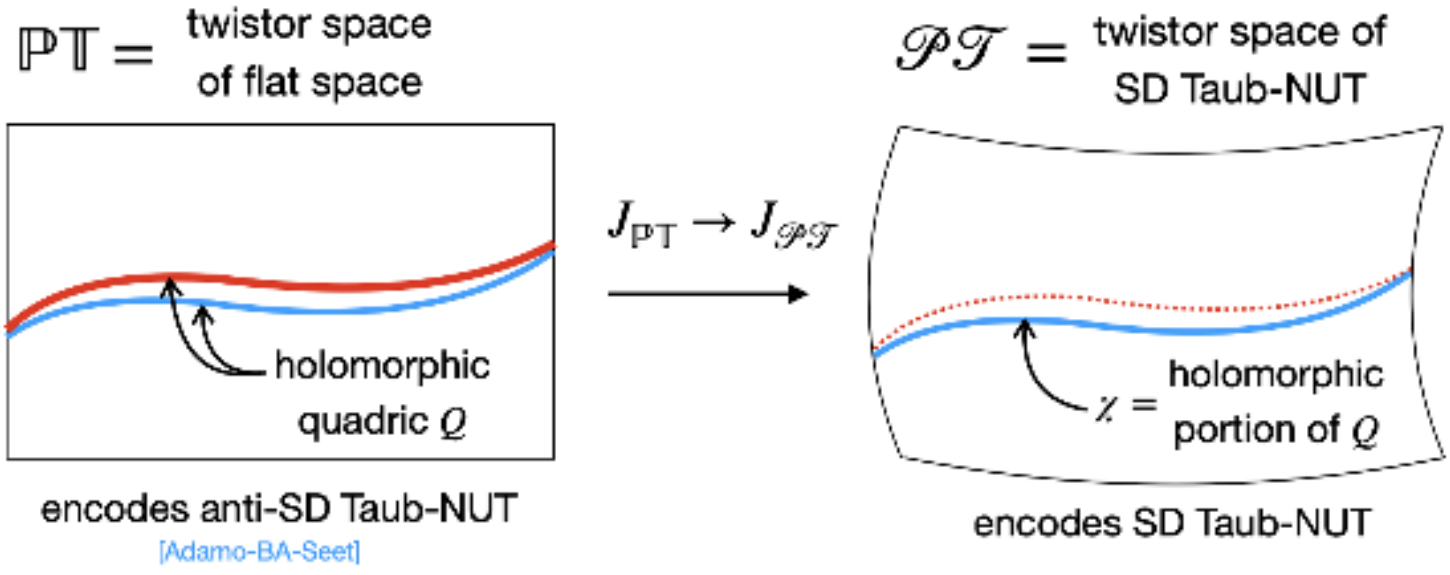
[Adamo-BA-Seet]

$J_{\mathbb{PT}} \to J_{\mathscr{PT}}$

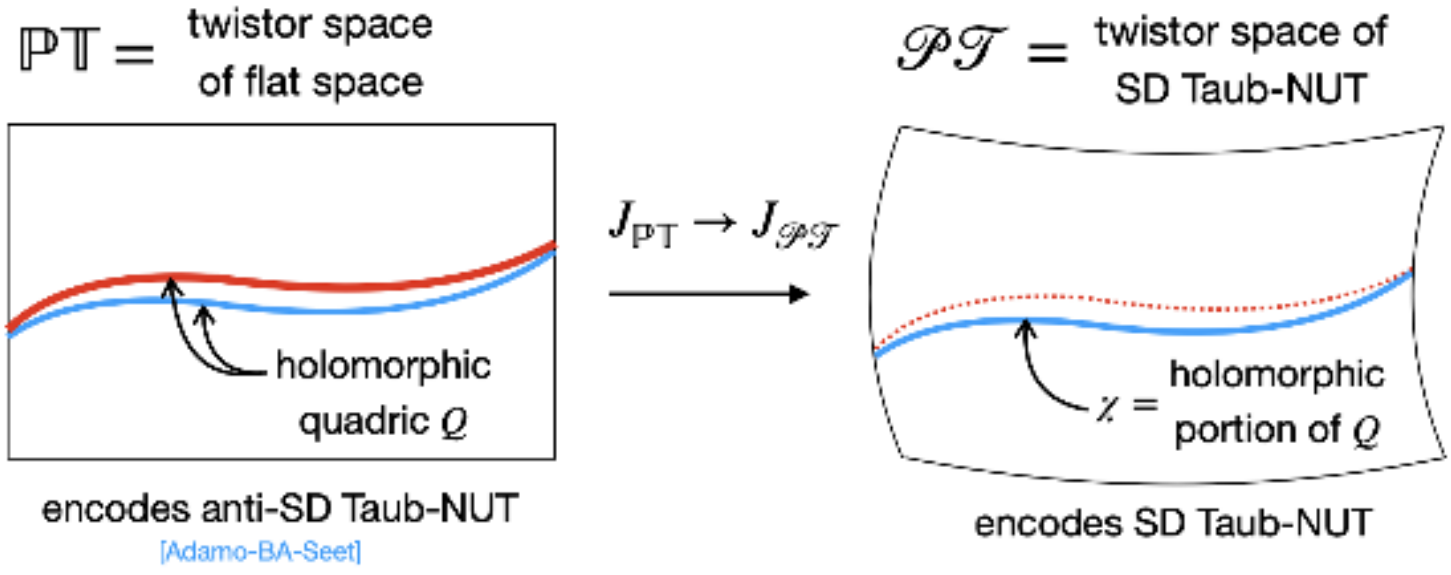
$\mathscr{PT}$ = twistor space of SD Taub-NUT



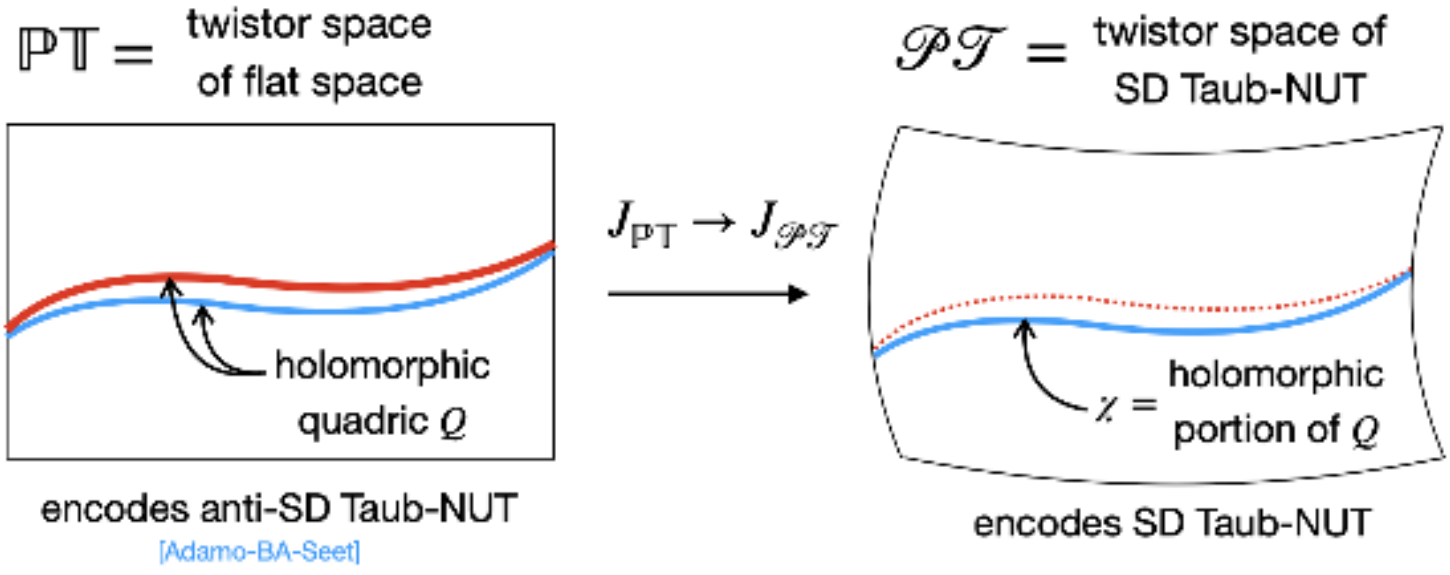
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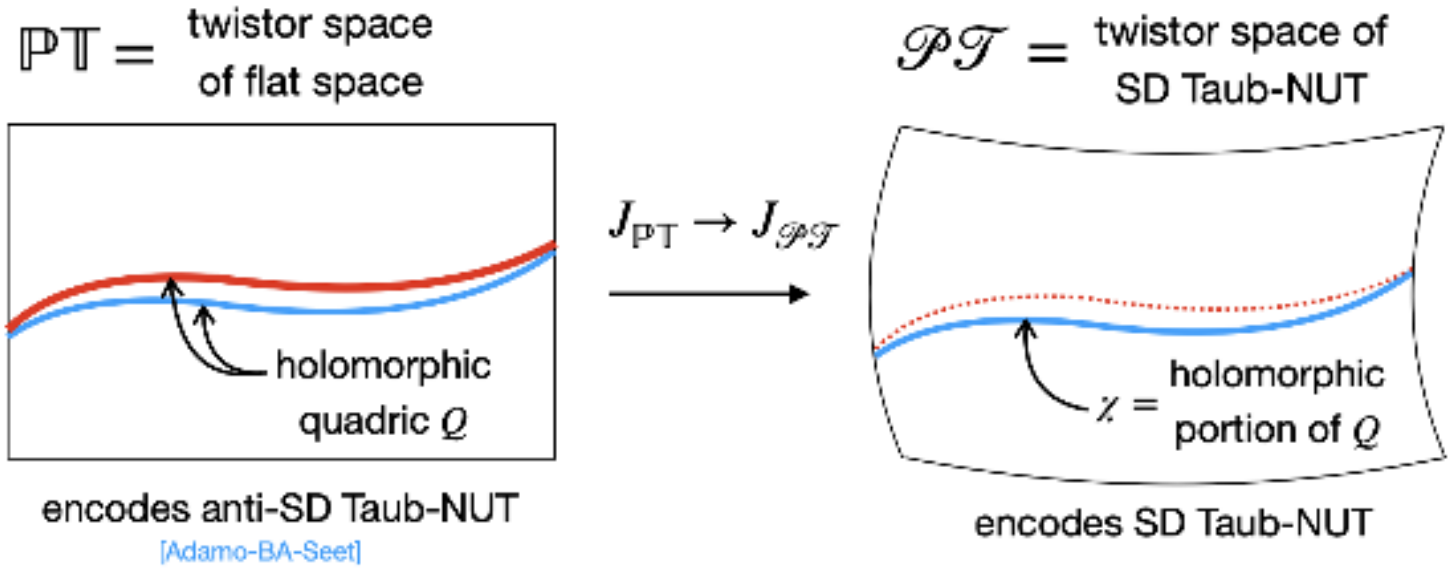


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- **Theorem** [Adamo-BA-Seet-Sharma 2026]: $g =$ Schwarzschild

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Thank you!