Arithmetic Dynamics Week 5

Dynamics over Local Fields: Good Reduction II.

Motivation: $\phi_a : [aX^d, Y^d]$

$$\tilde{\phi}_a = \begin{cases} [\tilde{a}X^d, Y^d], & \text{if } v(a) = 0 \\ [0,1], & \text{if } v(a) > 0 \\ [1,0], & \text{if } v(a) < 0. \end{cases}$$

Only in the first case is the reduced map interesting.

$(K, |\cdot|_v)$ nonarchimedean valued field

$R = \mathcal{O}_K$ ring of integers of K

$R^\times$ unit group

$\mathfrak{p}$ the unique maximal ideal of R

$k = R/\mathfrak{p}$ the residue field

$\tilde{a}$ is the image of $a \in R$ under the projection $R \to R/\mathfrak{p}$.

Theorem: Let $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map defined over K and write $\phi = [F, G]$ in normalised form.

[i.e. $F, G \in R[X, Y]$, minimum valuation of coeffs of F and G is 0]

Then TFAE:

(a) $\deg(\phi) = \deg(\tilde{\phi})$

(b) $\tilde{F}(X, Y)$ and $\tilde{G}(X, Y)$ have no common roots in $\mathbb{P}^1(\overline{k})$

(c) $\mathrm{Res}(\phi) \in R^\times$ [$\mathrm{Res}(\phi) := \mathrm{Res}(F, G)$]

(d) $\mathrm{Res}(\tilde{F}, \tilde{G}) \neq 0$.

Proof: (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) clear by properties of $\mathrm{Res}(A, B)$:

[(c) means image of $\mathrm{Res}(\phi)$ is nonzero in k]

Observe that $\mathrm{Res}(\tilde{F}, \tilde{G}) = \widetilde{\mathrm{Res}(F, G)}$.

We also have

$$\deg(\tilde{\phi}) = \deg(\phi) - (\text{number of common roots of } \tilde{F} \text{ and } \tilde{G}).$$ □

Definition: $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K is said to have **good reduction** if it satisfies any of the equivalent conditions of the previous theorem.

Theorem: $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ good reduction. Then

(a) ϕ is everywhere non-expanding, i.e.

$$\rho_v(\phi(P_1), \phi(P_2)) \leq \rho_v(P_1, P_2)$$

for all $P_1, P_2 \in \mathbb{P}^1(K)$.

(b) ϕ has empty Julia set.

Proof: (a) $\rho_v(\phi(P_1), \phi(P_2)) \leq \dfrac{\rho_v(P_1, P_2)}{|\mathrm{Res}(\phi)|_v^2}$

good reduction means $\longrightarrow$ $\mathrm{Res}(\phi) \in R^\times$, so $|\mathrm{Res}(\phi)|_v = 1$.

(b) (a) $\Rightarrow$ ϕ is equicontinuous everywhere. $\square$

Theorem 2.18: $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ good reduction. Then

(a) $\widetilde{\phi}(\tilde{P}) = \widetilde{\phi(P)}$ for all $P \in \mathbb{P}^1(K)$

(b) If $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ is another rational map with good reduction, then $\phi \circ \psi$ has good reduction, and

$$\widetilde{\phi \circ \psi} = \tilde{\phi} \circ \tilde{\psi}.$$

Proof: (a) Write $\phi = [F, G]$ in normalised form with $F, G \in R[X, Y]$ homogeneous. Write $P = [\alpha, \beta]$ in normalised form, $\alpha, \beta \in R$.

By good reduction [equivalence (b)], at least one of $F(\alpha, \beta)$ and $G(\alpha, \beta)$ is in $R^\times$, so that

$$\phi(P) = [F(\alpha, \beta), G(\alpha, \beta)]$$

is already in normalised form. Hence we have

$$\widetilde{\phi(P)} \overset{\text{def}}{=} [\widetilde{F(\alpha, \beta)}, \widetilde{G(\alpha, \beta)}]$$

$$= [\tilde{F}(\tilde{\alpha}, \tilde{\beta}), \tilde{G}(\tilde{\alpha}, \tilde{\beta})]$$

($R \to k$ is a ring homomorphism)

$$= \tilde{\phi}(\tilde{P}).$$

(b) Write $\phi = [F, G]$, $\psi = [f, g]$ in normalised form, and write

$$A = F \circ f, \quad B = G \circ g, \quad \phi \circ \psi = [F \circ f, G \circ g] = [A, B].$$

$A, B \in R[X, Y]$.

~~Suppose for contradiction that ϕ and ψ have good reduction.~~

Suppose, for contradiction, that $\overline{A}$ and $\overline{B}$ have a common root in $\mathbb{P}^1(\overline{k})$.

Then $\tilde{F}$ and $\tilde{G}$ have a common root $[\tilde{f}(\alpha, \beta), \tilde{g}(\alpha, \beta)]$ in $\mathbb{P}^1(\overline{k})$,

but ϕ has good reduction, so we must have $\tilde{f}(\alpha, \beta) = \tilde{g}(\alpha, \beta) = 0$,

contradicting that ψ has good reduction.

In particular $\tilde{A}$ and $\tilde{B}$ are not identically 0, so $\phi \circ \psi = [A, B]$ is normalised, so we have

$$\widetilde{\phi \circ \psi} = [\tilde{A}, \tilde{B}] = [\widetilde{F(f,g)}, \widetilde{G(f,g)}]$$

$$= [\tilde{F}(\tilde{f}, \tilde{g}), \tilde{G}(\tilde{f}, \tilde{g})]$$

$$= \tilde{\phi} \circ \tilde{\psi}. \quad \square$$

Remark: The converse of (b) is false:

it is possible for $\phi \circ \psi$ to have good reduction but both ϕ and ψ have bad reduction.

E.g: $\phi([x,y]) = [x^2, py^2]$

$\psi([x,y]) = [p^2x^2, y^2]$ over $\mathbb{Q}_p$

$\tilde{\phi} = [1,0]$, $\tilde{\psi} = [0,1]$

$(\psi \circ \phi)([x,y]) = [p^2x^4, p^2y^4] = [x^4, y^4]$.

Corollary: $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ rational map with good reduction. Then the reduction map sends periodic points to periodic points, and preperiodic points to preperiodic points.
Furthermore, if $P \in \mathrm{Per}(\phi)$ of exact period n and $\tilde{P} \in \mathrm{Per}(\tilde{\phi})$ has exact period m, then m divides n.

Proof: $P = \phi^n(P)$. Reduce both sides:

$$\tilde{P} = \widetilde{\phi^n(P)} = \tilde{\phi}^n(\tilde{P}),$$

so $\tilde{P}$ is periodic. (similar for preperiodic)

Let m be the exact period of $\tilde{P}$ and write $n = mk + r$, $0 \le r < m$. Then

$$\tilde{P} = \tilde{\phi}^n(P) = \tilde{\phi}^r \circ \underbrace{\tilde{\phi}^m \circ \cdots \circ \tilde{\phi}^m}_{k \text{ times}}(\tilde{P})$$

$$= \tilde{\phi}^r(\tilde{P})$$

Minimality of $m \Rightarrow r = 0$, so $m \mid n$. $\square$

Theorem: $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ rational function of degree $d \ge 2$ over a local field $(K, |\cdot|_v)$. Suppose ϕ has good reduction and $P \in \mathbb{P}^1(K)$ is a periodic point. Define:

n = exact period of P for ϕ

m = exact period of $\tilde{P}$ for $\tilde{\phi}$

r = the order of $\lambda_{\tilde{\phi}}(\tilde{P}) := (\tilde{\phi}^m)'(\tilde{P})$ in $k^\times$

p = $\mathrm{char}(k)$

Then we have:

$n = m$, or $n = mr$, or $n = mrp^e$ (some $e \ge 1$).

Remark: We say an elliptic curve E/K has good reduction if the reduction $\tilde{E}/k$ is an elliptic curve.
In the case $[K:\mathbb{Q}_p] < \infty$, we have that

$$E(K)[n] \hookrightarrow \tilde{E}(k)[n] \quad \text{if } p \nmid n.$$

If $P \in E(K)$ is a torsion point of exact order n and $\tilde{P} \in E(k)$ has exact order m, then $n = m$ or $n = mp^e$ for some $e \geq 1$.

Proposition: $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ rational map with good reduction. Then every periodic point of ϕ is nonrepelling.

Proof: Let P be a periodic point of exact period n. Since ϕ has good reduction, so does ϕ^n.
By using $PGL_2(R)$, we may assume $P = [0,1]$

[If $P = [\alpha, \beta]$, $\alpha, \beta \in R$ in normalised form, at least one of α and β is a unit.
If $\alpha \in R^\times$, apply $\begin{pmatrix} \beta & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix}$
If $\beta \in R^\times$, apply $\begin{pmatrix} \beta & -\alpha \\ 0 & \beta^{-1} \end{pmatrix}$]

Since P is fixed by ϕ^n, we can write

$$\phi^n(z) = \frac{F(z)}{G(z)} = \frac{a_1 z + \cdots + a_d z^d}{b_0 + b_1 z + \cdots + b_d z^d}.$$

in normalised form.

Since ϕ^n has good reduction and $\widetilde{F}(0) = 0$, we must have $b_0 \in R^\times$.
[otherwise 0 would be a common root of $\widetilde{F}$ and $\widetilde{G}$]

Hence, we have

$$\lambda_\rho(\phi) = (\phi^n)'(0) = \frac{a_1}{b_0} \in R,$$

(just compute)

so $|\lambda_\rho(\phi)|_v \leq 1$, as required. □

Example: $\phi(z) \in \mathbb{Z}_2[x]$, $\deg \phi \geq 2$, leading coefficient of ϕ is in $\mathbb{Z}_2^\times$, so that ϕ has good reduction.
Let $P \in \mathbb{P}^1(\mathbb{Q}_2)$ be a periodic point of exact period $n \geq 2$.
Then n is of the form mrp^e, where m is the period of $\widetilde{P}$ in $\mathbb{P}^1(\mathbb{F}_2)$.
Since $\mathbb{P}^1(\mathbb{F}_2)$ only has 3 elements and the point at infinity cannot be in the orbit of $\widetilde{P}$ under a polynomial, we have $m \in \{1, 2\}$.
Also, since $\mathbb{F}_2^\times = \{1\}$, $r = 1$ is forced.
Hence n must be a power of 2.

Proposition: $\phi(z) \in K(z)$, $\deg(\phi) = d \geq 2$ good reduction.

(a) $P \in \mathbb{P}^1(K)$ point of period n for ϕ. Then
$v(\phi^i(P), \phi^j(P)) = v(\phi^{i+k}(P), \phi^{j+k}(P))$
for all $i, j, k \in \mathbb{Z}$
[use periodicity to define $\phi^i(P)$ for $i < 0$]

(b) $P \in \mathbb{P}^1(K)$ exact period n. Then
$v(\phi^i(P), \phi^j(P)) = v(\phi(P), P)$
for all $i, j \in \mathbb{Z}$ s.t. $\gcd(i-j, n) = 1$.

(c) $P_1, P_2 \in \mathbb{P}^1(K)$ periodic points for ϕ with exact periods n_1, n_2 resp. s.t. $n_1 \nmid n_2$ and $n_2 \nmid n_1$. Then $v(P_1, P_2) = 1$.

Corollary: $\phi \in K(z)$ as above, $P_1, P_2 \in \mathbb{P}^1(K)$ periodic of exact periods $n_1, n_2, n_1 \nmid n_2, n_2 \nmid n_1$. Write $P_i = [x_i, y_i]$ in normalised form. Then $x_1 y_2 - x_2 y_1 \in R^\times$.

Proof: By part (c), we have

$$1 = v(P_1, P_2) \overset{\text{normalised}}{=} |x_1 y_2 - x_2 y_1|_v. \quad \square$$

[analogy with cyclotomic units $\frac{\zeta^i - 1}{\zeta - 1}$ in $\mathbb{Q}(\zeta)$

Theorem: $\phi(z) \in R[z]$, $\deg(\phi) = d \geq 2$, leading coefficient is a unit in R.
Let $\alpha \in K$ be a periodic point of ϕ with exact period $n \geq 2$, $i, j \in \mathbb{Z}$, $\gcd(i-j, n) = 1$
Then

$$\frac{\phi^i(\alpha) - \phi^j(\alpha)}{\phi(\alpha) - \alpha} \in R^\times.$$

Remark: We can also define good reduction at a prime of a global field.

Case $K=\mathbb{Q}$:

$\phi(z) \in \mathbb{Q}(z)$ is in normalised form if

$$\phi(z) = \frac{F(z)}{G(z)} = \frac{a_0 + a_1 z + \cdots + a_d z^d}{b_0 + b_1 z + \cdots + b_d z^d}$$

for some $a_i, b_i \in \mathbb{Z}$ such that $\gcd(a_0, a_1, \ldots, a_d, b_0, \ldots, b_d) = 1$.

The reduction of ϕ modulo p is

$$\tilde{\phi}(z) = \frac{\tilde{F}(z)}{\tilde{G}(z)} \in \mathbb{F}_p(z).$$

We say ϕ has good reduction at p if $\deg(\phi) = \deg(\tilde{\phi})$.