Talk 2: Rafaelan Ioannou 20/10/2026

Asymptotic Cones

Part I: Ultralimits continued, asymptotic cones and the Morse Lemma.

Recall: Last time we introduced ultralimits in topological spaces, as well as ultralimits of sequences of metric spaces.

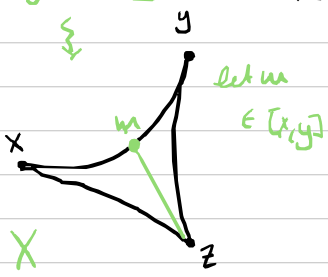
We will expand a bit on the latter by considering, specifically, CAT(0) metric spaces.

Note: this will be a relatively quick example, as we will not use the CAT(0) property in the rest of the reading group.

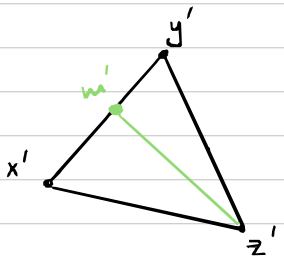
Defn (CAT(0)): A geodesic metric space is CAT(0) if geodesic triangles are thinner than their Euclidean counterparts.

More specifically:

geodesic Δ



let $m \in [x,y]$ in $\mathbb{E}^2$:



$d_E(x',y') = d_X(x,y)$
$d_E(y',z') = d_X(y,z)$
$d_E(z',x') = d_X(z,x)$

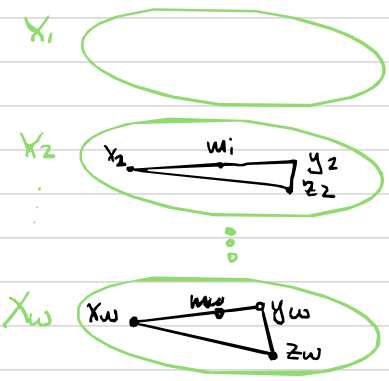
let m': $d_E(y',m') = d_X(y,m)$

X is CAT(0) if $d_X(m,z) \leq d_E(m',z')$.

$\rightsquigarrow$ CAT(0) spaces are non-positively curved, eg hyperbolic spaces are CAT(0).

Ultralimits × CAT(0): Ultralimits of sequences of CAT(0) spaces are CAT(0).

Why? Recall: sequence of m.s.



Let's see what happens to triangles in X_ω:
The 4 points correspond to sequences $x_n, y_n, z_n, m_n \in X_n$.

$\rightsquigarrow d_\omega((x_n),(y_n)) = \lim_\omega d_n(x_n, y_n) \in [0,\infty]$

ω such that $d_n(x_n, y_n) \in U$ for every nhood U of ω.

$\rightsquigarrow X_\omega$: sequences st. this distance is finite. $(x_n) \sim (y_n)$ $d_\omega = 0$.

$\rightsquigarrow$ We assumed all X_i are CAT(0).
Hence, $\forall i$: the corresponding points x_i, y_i, z_i form a triangle. Let $m_i \in [x_i, y_i]$.
Then $d_{X_i}(m_i, z_i) \le d_{\mathbb{E}^2}(m_i', z_i')$.
(m_n) corresponds to the point m_ω and $d_\omega((m_n),(z_n)) = \lim_\omega d_n(m_n, z_n) \le$ $\mathbb{E}^2$ dist.

Asymptotic Cones

Let (X,d) be a metric space. Let (λ_n) be a sequence of positive scale factors λ_n s.t. $\lim_\omega \lambda_n = 0$, and set $d_n := \lambda_n d$.

$\rightsquigarrow$ The sequence (X, d_n) consists of rescaled copies of (X,d)

Defn: An asymptotic cone of X, $\mathrm{Cone}(X)$ or $C(X)$, is the ultralimit of the sequence of pointed metric spaces: $C(X) = \lim_\omega (X, \lambda_n, p_n)$

meaning, the space with the "observers" as Talia defined.

* Warning! This depends on the choice of ω, λ_n, p_n.

Lemma/Eg: Let X be $\mathbb{H}^k$, $k \geq 2$.

Every asymptotic cone $X_\omega = C(X)$ is a tree.

Specifically: a tree that branches at every point and has infinite degree of branching at every pt x_ω.

Proof: How do we prove something is a tree?

It suffices to show that triangles are 0-slim.

We use CAT(0) of X to assert that every geodesic is unique. Hyperbolic spaces have δ-slim triangles. In particular here $\delta < 1$.

Hence triangles in X_ω are $\lambda_n \delta$-slim. So $\lim_\omega \lambda_n \delta = 0$ since $\lim_\omega \lambda_n = 0$. □

Quasi-isometries:

Let's see what happens if we have a quasi-isometric embedding: $X \to X'$

$$\frac{1}{L} d(x,y) - A \leq d(f(x), f(y)) \leq L\, d(x,y) + A$$

Pick a sequence of scale factors λ_n, observers $p_n \in X$ and $q_n = f(x_n)$

$$\frac{\lambda_n}{L} d(x,y) - \lambda_n A \leq \lambda_n d(f(x), f(y)) \leq L \lambda_n d(x,y) + \lambda_n A$$

Distances: $d_{x_n} = \lambda_n d_X$, $d_{x_n'} = \lambda_n d_{X'}$

$$\frac{1}{L} d_{x_n}(x,y) - \lambda_n A \leq d_{x_n'}(f(x), f(y)) \leq L d_{x_n}(x,y) + \lambda_n A_n.$$

Taking the ultralimit: $f_\omega : X_\omega \to X'_\omega$ $\quad f_\omega((x_n)) = (f(x_n))$

$$\frac{1}{L} d_\omega(x,y) \leq d_\omega(f_\omega(x), f_\omega(y)) \leq 1\, d_\omega(x,y)$$

Bi-Lipschitz, w/ no additive constant!

Also, if f was q-surjective, f_ω is surjective $\therefore$ a homeomorphism.

Direct example: $\mathbb{R}^n \overset{q.i.}{\simeq} \mathbb{R}^m \iff n = m$.

* not useful for $\mathbb{H}^n$!

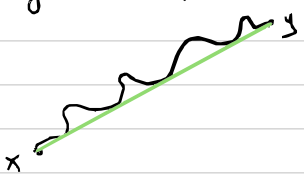
Key: $C(\mathbb{R}^n) = \mathbb{R}^n$

If not, this would induce a bi-Lipschitz homeomorphism between the asymptotic cones. But we cannot have $\mathbb{R}^n \overset{\text{hom}}{\simeq} \mathbb{R}^m$ for $n \neq m$!

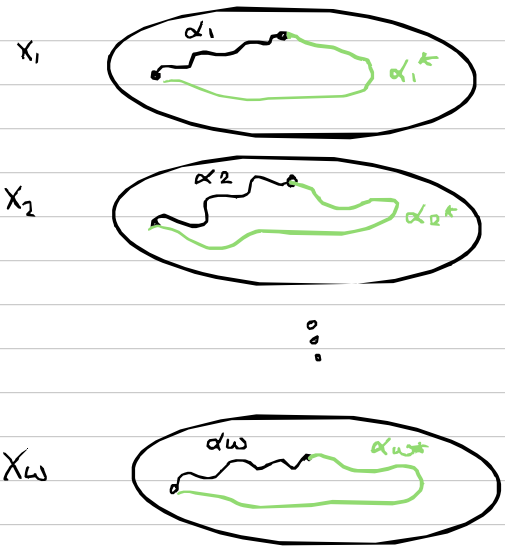
Morse Lemma (Quasi-geodesic stability)

Setup: $X = \mathbb{H}^n$. We say f is a q.geodesic if it is a q.i. embedding $f: J \to X$ s.t. J is an interval in $\mathbb{R}$.

Morse Lemma: "Q-geodesics remain close to geodesics"
$\exists$ fn. $D(L,A)$ s.t. every (L,A)-q.geodesic α in X is D-Hausdorff close to a geodesic α^*



Proof: Idea: If the lemma is not true, then we get a sequence of counter-examples, ie q.geods α_i. That yields a bi-Lipschitz map from our interval to asymptotic cone X_ω, in the same way as before.
The cone of $\mathbb{H}^n$ is a tree, so bi-Lipschitz maps are geodesics!
Say this is a geodesic α_ω^*.
The sequence of geodesics α_i^* that connect endpoints of the q.geods. α_i converge to some geodesic in the cone. This has the same endpoints as α_ω^*, and X_ω is uniquely geodesic, so they define the same geod.
Hence the distance between q.geods α_i and geods α_i^* cannot diverge to infinity by defn of ultralimit.



each α_i is a counterexample. Hence let α_i^* be a geod. with the same endpoints
& the sequence (α_i^*) converges to α_ω^*

★ Remark: By q.i. and asymp.-cones, the q.geod. α_ω is actually bi-Lipschitz. In $C(\mathbb{H}^n)$ that is a geodesic!

uniquely geodesic