QI Rigidity Reading Group – 27/10/2026

Boundary Extension and Quasi-conformal Maps

Def (1.10): A geodesic in a metric space (M,S) is a distance preserving map γ of an interval $I \subset \mathbb{R}$ to X

A geodesic ray is a geodesic whose domain is a half line $[0,\infty)$.

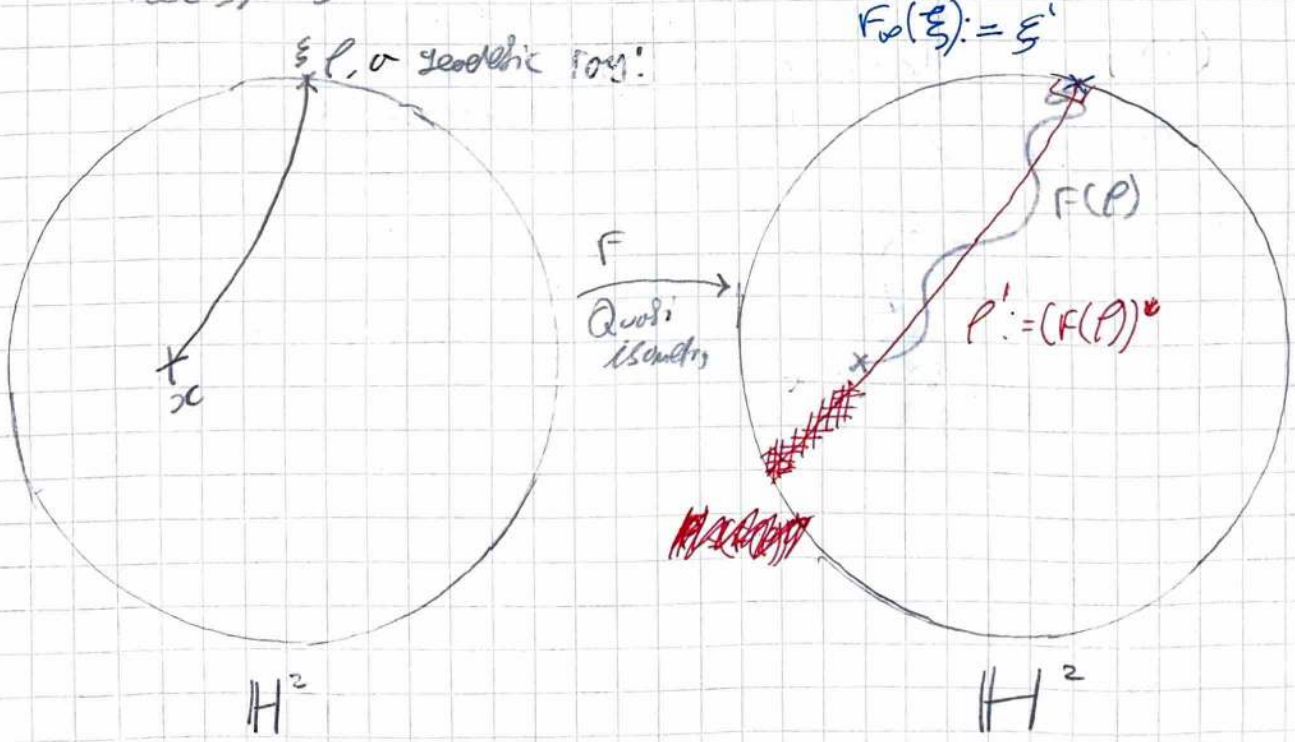
Notation: * for like γ^* or α^* indicates strictly a geodesic as we are now working with quasigeodesics, or an "induced" geodesic

Recall: let $X = \mathbb{H}^n$, $F: X \to X$ a QI map. Then by the Morse lemma, F sends geodesic rays uniformly close to geodesic rays. That is $\forall \rho, \exists \rho'$ s.t. $d(F(\rho), \rho') \in D$ where ρ & ρ' are geodesic rays. Moreover $\rho' = (F(\rho))^*$

D is the Hausdorff Distance & a function of L, A

Hausdorff = set distance Distance

Let ξ, ξ' be the limit of the rays ρ, ρ' on the boundary sphere of $\mathbb{H}^n$. Then we set $F_\infty(\xi) = \xi'$



- $\lim_{t \to \infty} \rho(t) \in S^{n-1} = \partial \mathbb{H}^n$

- somewhat not obvious fact: ξ' only depends on ξ and not the choice of ρ that limits to ξ

• Note: This lets us define our QI F on the boundary. i.e, we get a boundary extension

Taking the boundary extension plays nicely with the structure of QI

That is we define it by the limiting behaviour of F on the boundary.

• Exercise/Facts: 1) $(FG)_\partial = F_\partial \circ G_\partial$ $\forall$ $F, G: X \to X$, quasi-isometries

Boundary Extension also preserves the semi-group structure of QI (x) or ~~QI (H)~~

2) Suppose that $d(F,G) \leq \infty$, i.e. $\exists C < \infty$ s.t. $d(F(x), G(x)) \leq C$ $\forall x$. Then $F_\partial = G_\partial$

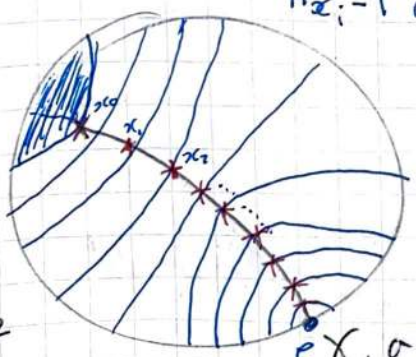
3) If $\bar{F}$ is quasi-inverse to F, then $(\bar{F})_\partial$ is " " F_∂

• Def/Exposition: Let γ be a geodesic ray in $\mathbb{H}^n$ and $\pi = \pi_\gamma : \mathbb{H}^n \to \gamma$ be the orthogonal projection. Then $\forall x \in \gamma$ (Except the initial point), $H_x := \pi^{-1}(x)$ is an $n-1$ dimensional subspace of $\mathbb{H}^n$.

Using our extension theorem we get π extends to a projection onto the boundary

$$\pi : \mathbb{H}^n \cup S^{n-1} \setminus \{\xi\} \longrightarrow \gamma$$

H_{x_1} = 1 dimensional subspace
H_{x_0} = 2 dimensional subspace



$\mathbb{H}^2$

ξ γ, σ geodesic rays starting at x_0

Comp-3.43: • Quasi-isometries quasi-commute with the nearest point projections. That is:

Let $F: \mathbb{H}^n \to \mathbb{H}^n$ be an (L,A)-QI. Let γ be a geodesic ray, γ' a geodesic ray within distance $\leq D(L,A)$ from the quasi-geodesic $F(\gamma)$. Let $\pi: \mathbb{H}^n \to \gamma$, $\pi': \mathbb{H}^n \to \gamma'$ be nearest point projections.

Then, for some $C = C(L,A)$ we have

$$d(F\pi, \pi' F) \leq C$$

i.e.

$$\forall x \in \mathbb{H}^n,\ d(F\pi(x), \pi' F(x)) \leq C$$

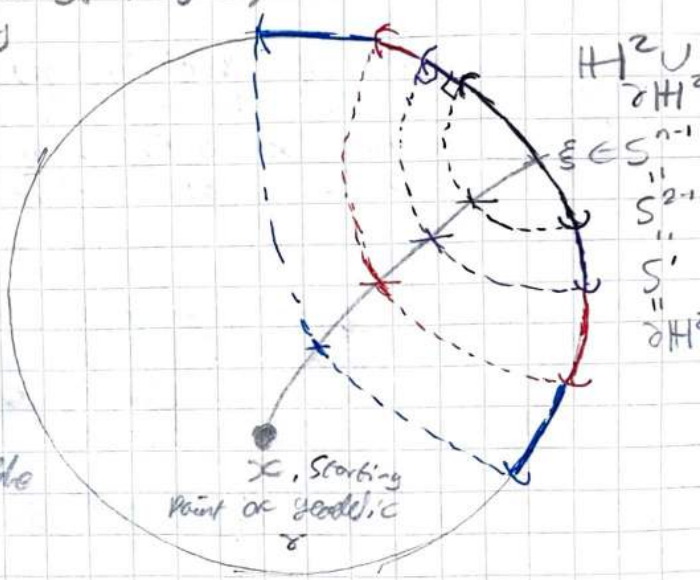
In short if we put this into a diagram

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{F} & \mathbb{H}^n \\ \downarrow \pi' & & \downarrow \pi \\ \gamma \subset \mathbb{H}^n & \xrightarrow{F} & \gamma' \subset \mathbb{H}^n \end{array}$$

clearly $F\pi \neq \pi' F$ $\forall \gamma$ but it is close to it

You can think that this diagram is "distance" from commuting.

Discussion: we can take ξ to be the limit of γ on the boundary. we can also define balls on the boundary of $\mathbb{H}^n$ by looking at where each H_{x_i} intersects the boundary. This gives us a basis for the topology at the point ξ.



Lets take a sequence of points $y_i = F(x_i)$ where F is a QI & $x_i \in \gamma$.
$|y_i| = d(0, y_i) - d(0, F(x_i))$
$\geq \frac{1}{L} d(F(0), x_i) - A$ but observe that as $i \to \infty$ $x_i \to \infty$ when we only look inside $\mathbb{H}^n$. So to achieve this we look at the boundary and see $\lim_{i \to \infty} y_i = \xi$

Pre-images of open sets are open

Then by the Lemma 3.43, we see that $F_\infty(B_i)$ are contained in round balls B_i' determined by the orthogonal projection onto the QI of F & then taking the pre-image. The intersection of these balls though is $\xi := F_\infty(\xi)$ and so F_∞ is continuous i.e. a homeomorphism!

Lemma 3.44: For every QI $f: \mathbb{H}^n \to \mathbb{H}^n$, the boundary extension F_∞ is a homeomorphism.

Note: the boundary extension is a seperate map to the QI, it is only a map of the boundary that is derived from the map f.

Corollary 3.45: $\mathbb{H}^n$ is QI to $\mathbb{H}^m$ iff $n = m$

§: Quasi-Actions

Recall: The action of a group G on a set X is a homomorphism $\phi: G \to Aut(X)$ s.t.

$$\phi(g_1 g_2) = \phi(g_1)\,\phi(g_2)$$

That is, the group structure restricts the automorphism structure

Definition: Let G be a group and X a metric space. A (L, A)-quasi action of G on X is a map $\phi: G \to \overline{QI}(X)$ s.t.

($\overline{QI}(X)$: semigroup of QI $X \to X$)

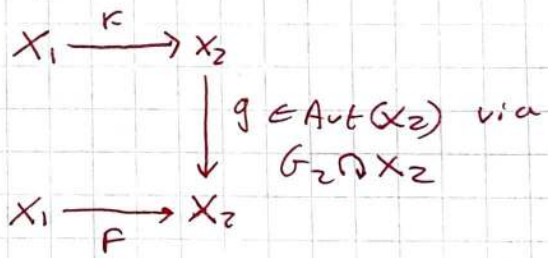
1. $\phi(g)$ is an (L,A)-QI of X $\forall g \in G$
2. $d(\phi(1), Id_X) \le A$
3. $d(\phi(g_1 g_2), \phi(g_1)\phi(g_2)) \le A \quad \forall g_1, g_2 \in G$

(2) & (3) can be interpreted as saying ϕ is almost a homomorphism with error A. Additionally, this error is constant across all ϕ.

Def: Suppose G_1, G_2 groups acting isometrically on metric spaces X_1, X_2, and $f: X_1 \to X_2$ is a QI with quasi-inverse $\bar{F}$. Then the conjugate quasi-action ϕ of G_2 on X_1 is

$$\phi(g) = \bar{F} \circ g \circ F$$

Quasi inverse $\bar{F}$ s.t.
$d(\bar{F}F(x), x) \le A$
$d(F\bar{F}(x'), x') \le A$



Ex: Suppose $X_1 = \mathbb{H}^n$, $\psi: G_1 \curvearrowright X$ is a geometric action and suppose that G_2 is a group QI to G_1. let $X_2 = G_2$. Then the QI $F: G_1 \to G_2$ gives a quasi-action $\phi_{F,\psi}$ of G_2 on $\mathbb{H}^n$.

Question: Can we extend the quasi action of a group on a set to the boundary?

Corollary 3.49: Every quasi-action ϕ of a group G on $\mathbb{H}^n$ extends (by $g \mapsto \phi(g)_\infty$) to an action ϕ_∞ of G on S^{n-1} by homeomorphisms

Lemma 3.50: $|\ker(\phi_\infty)| < \infty$.

Remark: Quasi-isometries & quasi-actions of $\mathbb{H}^n$ can be extended to homeomorphisms of $\partial\mathbb{H}^n = S^{n-1}$

Summary

Def: A Quasi action $\phi: G \curvearrowright X$ is

1) Properly Discontinuous if $\forall$ bounded subset $B \subset X$, $\{g \in G : \phi(g)(B) \cap B \neq \emptyset\}$ is finite

2) Cobounded if $\exists$ a bounded subset $B \subset X$ s.t. $\forall x \in X \ \exists g \in G$ s.t. $\phi(g)(x) \in B$

3) Geometric if properly discontinuous & Cobounded

§: Conical Limit points of Quasi Actions

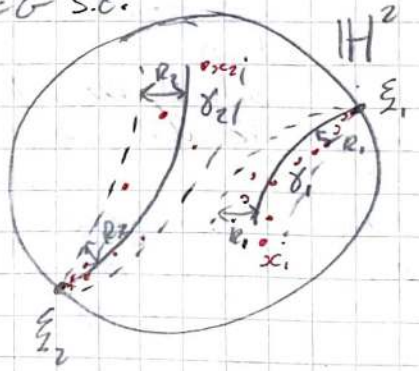
Def: Let ϕ be a quasiaction of G, a group on $\mathbb{H}^n$. A $\xi \in S^{n-1}$ is called a Conical limit point for ϕ if the following holds:
(equivalently every) ✓ (equivalently every) ✓

For some geodesic ray $\gamma \subset \mathbb{H}^n$ limiting to ξ and some point $x \in \mathbb{H}^n$, $\exists R < \infty$ and $(g_i) \in G$ s.t.

1) $\lim_{i \to \infty} \phi(g_i)(x) = \xi$

2) $d(\phi(g_i)(x), \gamma) \leq R \ \forall i$

See Picture



Lemma 3.52: Suppose $\Psi: G \curvearrowright X = \mathbb{H}^n$ is a Cobounded quasi-action. Then every point of the boundary sphere is a Conical limit point of Ψ.

Corollary 3.53: Suppose G is a group and $f: \mathbb{H}^n \to G$ is a QI. Then every point of S^{n-1} is a ~~Conical~~ Conical limit point of the Quasi-action $\Psi: G \curvearrowright \mathbb{H}^n$ which is conjugate to the action $G \curvearrowright G$ via f.

~~with every point on the boundary, we can find a γ, geodesic ray s.t. there is a sequence?~~

§ Quasiconformality of the Boundary Extension

From now on, we are in the UHP (upper half plane) model.

Recall: After compositions of f with isometries of $\mathbb{H}^n$ we get, and assume:

1) $\xi = 0 \in \mathbb{R}^{n-1}$, γ is the vertical geodesic above ξ

2) $0 = \xi' = F_\infty(\xi) \in \mathbb{R}^{n-1}$

3) $F_\partial(\infty) = \infty$. In particular, γ maps to a quasi-geodesic within bounded distance from the vertical geodesic $\gamma' = \gamma$ above $\xi' = \xi = 0$

Exposition: Let $F: \mathbb{H}^n \to \mathbb{H}^n$, (L,A)-QI:

1) Consider an Annulus $A \subset \mathbb{R}^{n-1}$ given by $A = \{x : R_1 \le |x| \le R_2\}$
2) R_2/R_1 is called the eccentricity
3) the length of $\pi_\gamma(A)$ in γ is $d = \log(R_2/R_1)$
4) Recall, F, quasi-commutes with the orthogonal projection
$\left(d(F \circ \pi_\gamma, \pi_\gamma \circ F) \le C = C(L,A)\right)$
5) So $\pi_\alpha(A)$ is an interval of hyperbolic length $\le C' := 2C + Ld + A$

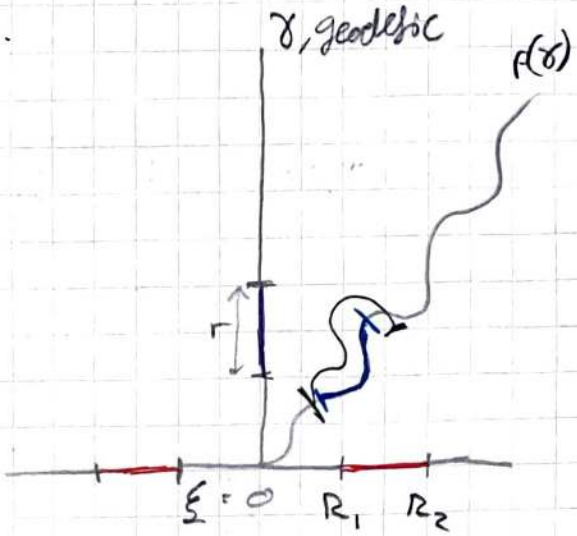
* α is not defined, either it is another geodesic which we then map to γ or ...

6) Here, $F(A)$ is contained in the euclidean annulus $A' :=$
$\{x : R_1' \le |x| \le R_2'\}$, $\frac{R_2'}{R_1'} \le e^{C'}$
7) $\eta(r) = e^{C'}$, $C' = 2C + L\log(r) + A \Rightarrow \eta(r) = r^L e^{2C+A}$

Which is continuous, monotonic $\therefore$

$$\lim_{r \to 1} \eta(r) = 1, \quad \lim_{r \to \infty} \eta(r) = \infty \ldots$$

we Proved (:)



$A = \{x : R_1 \le |x| \le R_2\}$
$= [-R_2, -R_1] \cup [R_1, R_2]$
$\pi_\gamma(A) = \{z \in \mathbb{H} : Re(z) = 0, |Im(z)| \in [R_1, R_2]\}$
$\pi_{F(\gamma)}(A)$
$F(\pi_\gamma(A))$

Lemma 3.5a: The topological annulus $F_\partial(A)$ is contained in an annulus A' s.t. the eccentricity of A' is $\le \eta(r)$ where r is the eccentricity of A.

In particular, round Spheres ($r=1$) map to Quasi-ellipsoids of eccentricity $\le e^{2C+A}$