QI Rigidity : Session 1

Plan for today:

I. Background
II. Examples
III. Ultralimits

I. Background

Before defining what is QI rigidity, we first need to know:

1. What is a quasi-isometry?
2. What is the Schwarz-Milnor Thm?
3. What is a virtual isomorphism?

(1) Quasi-isometries

- (Almost) every area of maths has its notion of equivalent objects
- In GGT, this is q.i.: preserving large-scale geometry.

Def: Let X, Y m.s. A map $f: X \to Y$ is called an (L, A)-quasi-isometry if:

(i) f is (L, A)-coarsely Lipschitz

$$d(f(x_1), f(x_2)) \leq L\, d(x_1, x_2) + A$$

(ii) There exists an (L, A)-coarse Lip. map $\bar{f}: Y \to X$ (quasi-inverse) s.t.

$$d(\bar{f}(f(x)), x) \leq A$$
$$d(f(\bar{f}(y)), y) \leq A$$

$L, A \in \mathbb{R}$

Spaces X, Y are q.i. if there exists an (L, A)-quasi-isometry $X \to Y$, for some L, A constants.

Intuitively: X, Y are q.i. if you squint and they look the same.

E.g.
- $\mathbb{Z} \overset{q.i.}{\sim} \mathbb{R}$
- X bounded m.s $\overset{q.i.}{\sim}$ $\{pt\}$
- Every metric space is q.i. to its metric completion.
  - ↳ Gromov hyp $X \overset{q.i.}{\sim} \partial X$ Gromov

Rmk: If you have seen q.i. before, you may have used a different definition, which turns out is equivalent to the one above:

Def 2 [QI]:

(i) qi. embedding

$$\frac{1}{L} d(x,y) - A \leq d(f(x), f(y)) \leq L d(x,y) + A$$

(ii) quasi-surjective

$$\exists C \text{ s.t. } \forall y \in Y, \exists x \in X \text{ s.t. } d(x,y) \leq C.$$

Note:
- $(L, 0)$-qi. is a bilipschitz homeo
- $(1, 0)$-qi. is an isometry

(2) Schwarz - Milnor Theorem

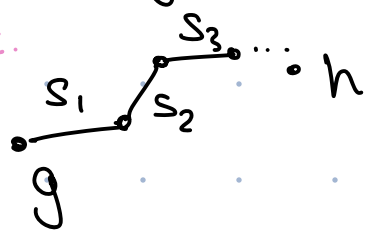
To motivate what we will see next, we introduce / recap Schwarz - Milnor.

Thm: G group, X m.s. s.t. $G \overset{\text{geo.}}{\curvearrowright} X$.
Then G is f.g. and $\forall x \in X$, the orbit map $g \mapsto g \cdot x$ is a q.i., where d_G is the word metric.

$\{g \in G : gK \cap K \neq \emptyset\}$ finite

geometric: proper, cocompact by isometries

word metric: $\exists K$ cpt. s.t. $\bigcup_{g \in G} gK = X$.



In other words:

nice group action + nice metric space $\Rightarrow$ G f.g. + $G \underset{\text{q.i.}}{\sim} X$

(3) Virtual isomorphisms

In view of Schwarz-Milnor, we have two more examples of q.i. groups:

E.g. $G' \underset{f.i.}{<} G \Rightarrow G \overset{q.i.}{\sim} G'$

E.g. $G' = G/F$, F finite $\Rightarrow G \underset{q.i.}{\sim} G'$.

Combining these examples, we obtain:

Def:

1. G_1 is VI to G_2 if there exist f.i. subgroups $H_i < G_i$ and finite normal subgrps $F_i \triangleleft H_i$ $(i = 1, 2)$ s.t. $H_1/F_1 \cong H_2/F_2$.
   ↳ a.k.a. commensurable
2. G is virtually cyclic if it is VI to a cyclic group.

Fact 1.: VI is an equivalence relation.

Fact 2. By Schwarz-Milnor examples above:

$$\underset{\text{algebra}}{\text{VI}} \Rightarrow \underset{\text{geometry}}{\text{QI}}.$$

Informally, q.i. rigidity is when VI $\Leftrightarrow$ QI.

Def:

1. A group G is QI rigid if every group G' which is QI to G, is infact VI to G.

2. A class $\mathcal{C}$ of groups is QI rigid if for every group G' which is QI to some $G \in \mathcal{C}$, there exists $G'' \in \mathcal{C}$ s.t. $G' \cong G''$.

3. A property P of groups is QI invariant whenever the class of groups satisfying P is q.i. rigid.

(II) Examples of QI rigidity

QI rigidity is special: we recover precise algebraic information from large-scale geometry, yet a lot of local information is ignored!

As demonstrated by the (huge!) list in the notes, many classes of groups are QI rigid, including:

E.g.
- Free groups
- π_1(closed, compact, w/o boundary surfaces/3-mfds)

- f.p. groups
- hyperbolic groups
- amenable groups
- ⋮

Non. e.g.:
- RF groups (For any $g \neq 1_G$, $\exists \phi: G \to F$ hom s.t. $\phi(g) \neq 1_F$)
- Simple groups
- Solvable groups
- Prop (T) not QI invariant

~ 10 minute break ~

(III) Ultralimits

Motivation: Quasi-isometries are not nice maps. For example, they need not be continuous (e.g. $\mathbb{R} \to \mathbb{Z}$, $x \mapsto \lfloor x \rfloor$). We use ultra limits to convert q.i. to homeomorphisms

Def: An ultrafilter on $\mathbb{N}$ is a measure $w: 2^{\mathbb{N}} \to \{0,1\}$ defined for all subsets of $\mathbb{N}$ s.t.

- Finitely-additive: $w(A \cup B) = w(A) + w(B) - w(A \cap B)$
- $w(\emptyset) = 0$
- $w(\mathbb{N}) = 1$

We say a subset E of $\mathbb{N}$ is w-large if $w(E) = 1$.

A property $P(n)$ holds for w-all $\mathbb{N}$ if $w(\{n : P(n) \text{ is true}\}) = 1$.

E.g. $P(n)$ = "n is not prime".

If $w(\{1, 4, 6, 8, 9 \dots\}) = 1$, then $P(n)$ holds for w-all $n \in \mathbb{N}$.

Def: An ultrafilter is trivial if $w(\{n\}) = 1$ for some $n \in \mathbb{N}$, and nontrivial if w vanishes on all finite sets.

From now on, all ultrafilters (unless otherwise stated) are non-trivial.

Def: X Hausdorff, w ultrafilter.
Then, for a sequence of points $(x_n) \in X$, we define the ultralimit (w-limit) $\lim_w x_n$ to be a point $a \in X$ s.t.
For every nhd U of a, $\{n \in \mathbb{N} : x_n \in U\}$ is w-large.

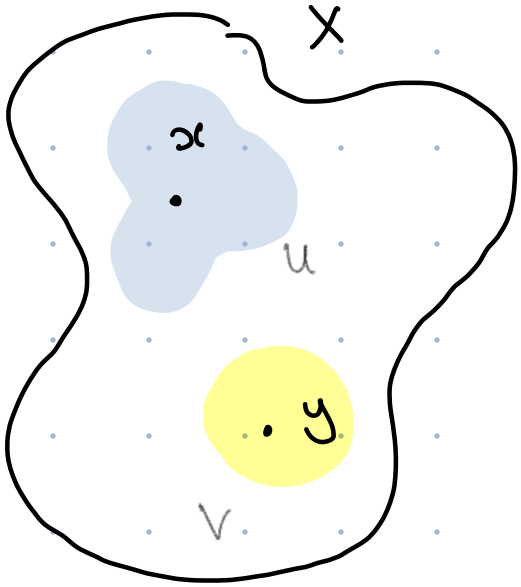
↪ i.e. $x_n \in U$ for w-all n.

Q: (John)
Does the converse of this lemma hold?

A: Yes!

Lem: As X Haus., a sequence $(x_n) \in X$ has at most one ultralimit.

Pf:
Suppose (x_n) has two distinct w-limits, $x \neq y$.
As X Hausdorff,
$\exists$ disjoint open sets U, V s.t. $x \in U$, $y \in V$.



Then,
$U' := w(\{n \in \mathbb{N} : x_n \in U\})$
$V' := w(\{n \in \mathbb{N} : x_n \in V\})$

$$w(U' \cup V') = w(U') + w(V') - w(U' \cap V')$$
$$= 1 + 1$$

$w(U' \cap V') = 0$

$\lightning$

Thm: If X compact, $(x_n) \in X$ has at least ultralimit.

Q: What is the ultralimit of $(-1)^n$ in $[-1, 1]$?

n	1	2	3	4	5
x_n	-1	1	-1	1	-1



Suppose $a \neq -1$ or 1.
For every nhd U of a, $\{n \in \mathbb{N} : x_n \in U\} = \emptyset$, hence not w-large. Thus a is either 1 or -1.

If $a = 1$, then for every nhd of 1,
$\{n \in \mathbb{N} \mid x_n \in U\}$ = even numbers $= E$

Similarly, if $a = -1$,
$\{n \in \mathbb{N} \mid x_n \in U\}$ = odd numbers $= O$

$$1 = w(\mathbb{N}) = w(O \cup E) = w(O) + w(E) - w(\emptyset)$$

Hence, exactly one of $w(O)$, $w(E)$ equals 1 whether $a = 1$ or $a = -1$ depends on the choice of ultrafilter!

Putting the previous two results together:

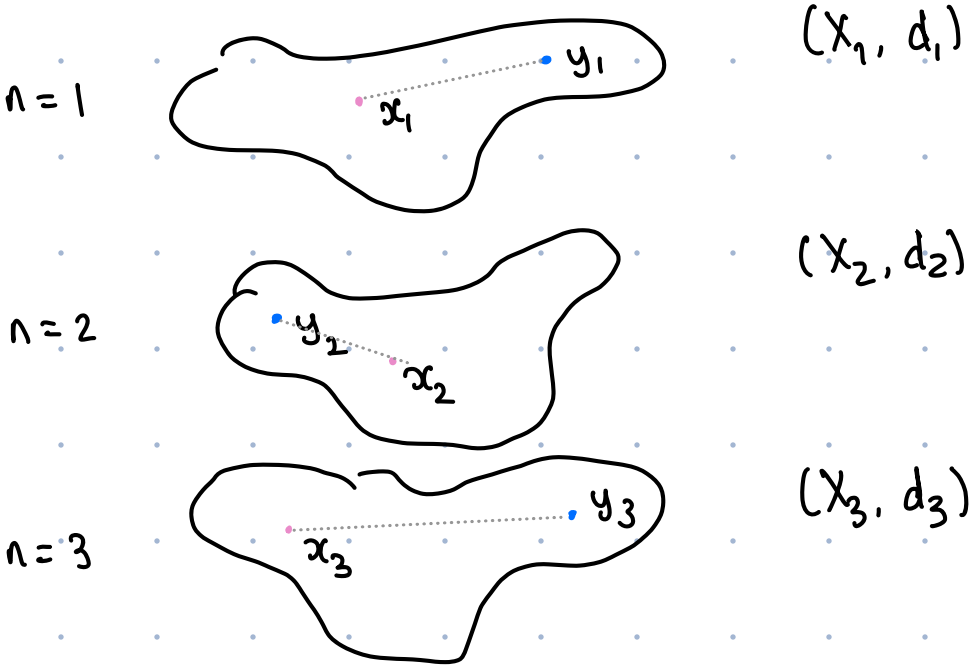
Cor: If X compact (and Haus.), then for any sequence $(x_n) \in X$, $\exists$ a unique ultralimit.

(III.2) Ultralimits of metric spaces

- Sequence of m.s. (X_n, d_n)
- Elts of ω-limit are equivalence classes of sequences $(x_n) \in X_n$.

Def: For every two sequences $(x_n) \in X_n$, $(y_n) \in X_n$:

$$d_\omega((x_n),(y_n)) := \lim_\omega d_n(x_n, y_n) \in [0, \infty)$$



Ex: d_w is a pseudometric. (relax $d(x,y)=0 \iff x=y$)

→ We want to take an equivalence relation over the sequences where $d_w = 0$.

→ Problem: What do we do when $d_w \to \infty$?

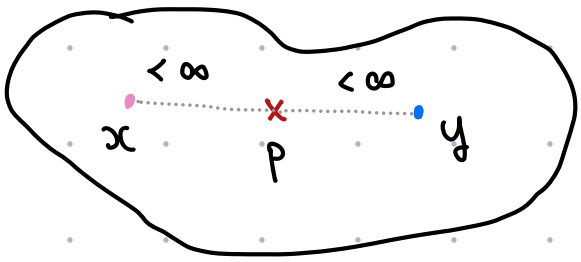
We introduce a sequence of observers $p_n \in X_n$.

Let $\mathcal{B} = \{(x_n) \mid d(x_n, p_n) < \infty\}$

Let X_w be the pointed ultralimit of a sequence of pointed m.s., i.e.

$$X_w = \mathcal{B}/\sim$$

where $(x_n) \sim (y_n)$ iff $\lim(d_w(x_n, y_n)) = 0$.



Limiting m.s. $\lim_w X$.

Comment (Dan):
The observers mean that (X_w, d_w) is not only pseudometric space, but a metric space. ‼

i.e. equivalence classes of sequences which observers can see.

Lem / Def:

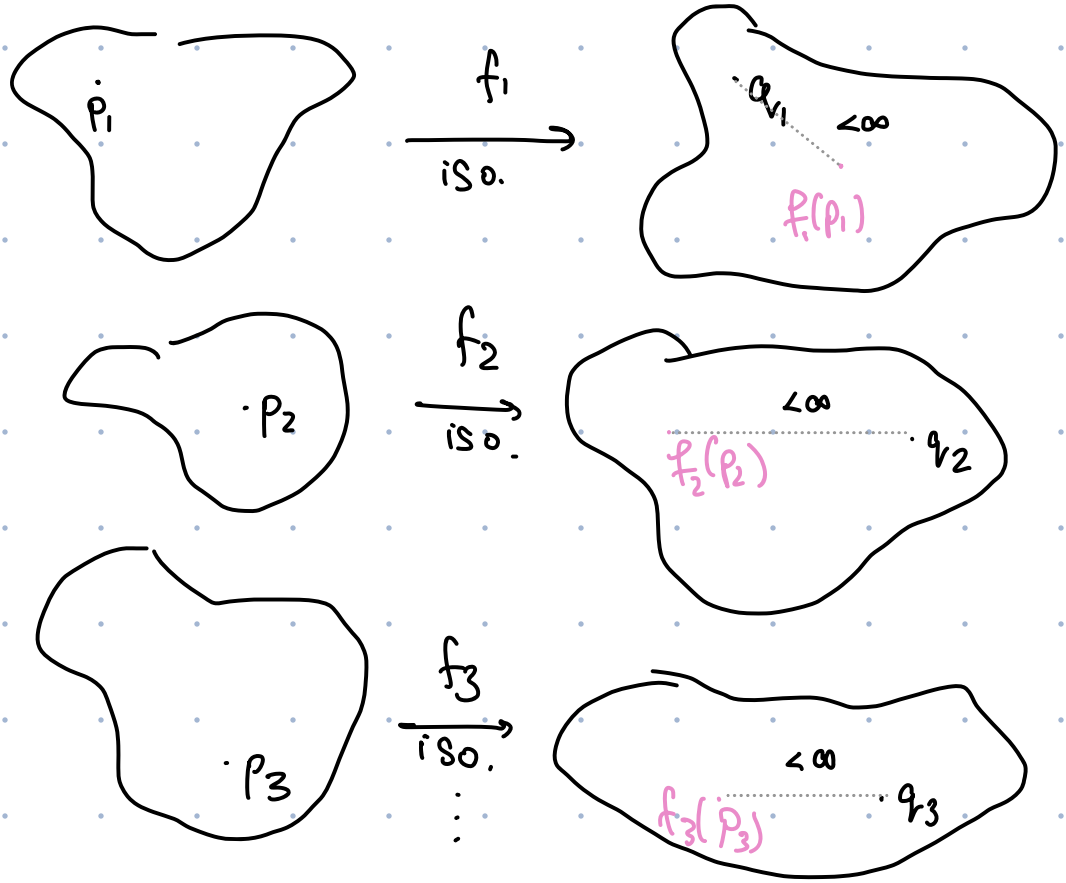
(X_n, p_n), (Y_n, q_n) pointed m.s.

$f_n : X_n \to Y_n$ isometries, so that

$$\lim_w d_{Y_n}(f_n(p_n), q_n) < \infty.$$

Then $f_w : X_w \to Y_w$ (equivalence classes of sequences)

$f_w(x_w) = (f_n(x_n))$ is an isometry.



(In particular, the ultralimit of a geodesic m.s. is a geodesic m.s.)