

Chapter 5 KNBS construction

Focus on completely positive mappings between C^* -algebras.

Lemma 5.1 If $\rho: A \rightarrow B$ is a positive linear map between C^* -algebras then ρ is bounded.

proof: Assume the contrary for A_+ , $\forall n \in \mathbb{N}$, $\exists a_n \in A_+$ s.t. $\|\rho(a_n)\| \geq n^3$

Now consider $c = \sum_{n=1}^{\infty} \frac{a_n}{n^2}$ (since $\sum_{n=1}^{\infty} \|\frac{a_n}{n^2}\| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, convergence)

Since A_+ closed, hence $c \in A_+$.

Assume $a_n \rightarrow a$, then $\|a - a^*\| \leq \|a - a_n\| + \|a_n - a^*\| = 2\|a - a_n\| \rightarrow 0$
 $\Rightarrow a = a^*$. Since $\exists R > 0$ s.t. $\|a_n\| \leq R$, hence $\sigma(a_n) \subseteq [0, R]$
 $\Rightarrow \|R1 - a_n\| = \max_{\lambda \in \sigma(a_n)} |R - \lambda| \leq R \Rightarrow \|R1 - a\| \leq R$
For $\lambda \in \sigma(a)$, since $a = a^*$, hence $R - \lambda \in \sigma(R1 - a)$
 $\Rightarrow |R - \lambda| \leq \|R1 - a\| \leq R \Rightarrow 0 \leq \lambda \leq 2R \Rightarrow \sigma(a) \subseteq [0, \infty) \Rightarrow a \in A_+$

Clearly $c \geq \frac{a_n}{n^2}$ ($\forall n \in \mathbb{N}$), hence $\rho(c) \geq \rho(\frac{a_n}{n^2}) = \frac{\rho(a_n)}{n^2} \geq 0$

$\Rightarrow \|\rho(c)\| \geq \frac{\|\rho(a_n)\|}{n^2} \geq n \Rightarrow$ contradiction since $\rho(c)$ bounded.

$\Rightarrow \rho$ bounded on A_+ . (Then for $a \in A_+$, just consider $\frac{a}{\|a\|} \in A_+$)

Since for $a \in A$, a can be decomposed into four positive elements, hence ρ bounded.

Lemma 5.2 let H be a Hilbert space and $p, q, t \in L(H)$ with $p \geq 0$ and $q \geq 0$.

Then (i) $\|t\| \leq 1$ iff $\begin{pmatrix} 1 & t \\ t^* & 1 \end{pmatrix} \geq 0$ in $L(H \oplus H)$

(ii) $\begin{pmatrix} p & t \\ t^* & q \end{pmatrix} \geq 0$ iff $|\langle \xi, t\eta \rangle|^2 \leq \langle \xi, p\xi \rangle \langle \eta, q\eta \rangle$ ($\xi, \eta \in H$)

(iii) if $\begin{pmatrix} p & t \\ t^* & q \end{pmatrix} \geq 0$, then $t^*t \leq \|p\|q$

Proof: (ii) " $\Rightarrow$ " For $\xi, \eta \in H$, $\langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} p & t \\ t^* & q \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rangle = \langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} p\xi + t\eta \\ t^*\xi + q\eta \end{pmatrix} \rangle$
 $= \langle \xi, p\xi \rangle + 2\operatorname{Re} \langle \xi, t\eta \rangle + \langle \eta, q\eta \rangle$

let $|\theta|=1$ s.t. $\langle \xi, \theta t\eta \rangle = |\langle \xi, t\eta \rangle|$ and replace η by $\lambda\theta\eta$ ($\lambda \in \mathbb{R}$)

Since $\langle \xi, \lambda\theta t\eta \rangle = \lambda |\langle \xi, t\eta \rangle|$ and $\langle \lambda\theta\eta, q(\lambda\theta\eta) \rangle = \lambda^2 \langle \eta, q\eta \rangle$,

hence $\langle \xi, p\xi \rangle + 2\lambda |\langle \xi, t\eta \rangle| + \lambda^2 \langle \eta, q\eta \rangle \geq 0$ ($\lambda \in \mathbb{R}$)

Now if $\langle \eta, q\eta \rangle > 0$, let $\lambda = -\frac{|\langle \xi, t\eta \rangle|}{\langle \eta, q\eta \rangle} \Rightarrow$ then $0 \leq \langle \xi, p\xi \rangle - \frac{|\langle \xi, t\eta \rangle|^2}{\langle \eta, q\eta \rangle}$
 $\Rightarrow |\langle \xi, t\eta \rangle|^2 \leq \langle \xi, p\xi \rangle \langle \eta, q\eta \rangle$

If $\langle \eta, q\eta \rangle = 0$, then $\langle \xi, p\xi \rangle + 2\lambda |\langle \xi, t\eta \rangle|^2 \geq 0$

if $|\langle \xi, t\eta \rangle| > 0$, let $\lambda \rightarrow -\infty$, then contradiction,

hence $|\langle \xi, t\eta \rangle| = 0$, and still have $0 \leq \langle \xi, p\xi \rangle \langle \eta, q\eta \rangle$

" $\Leftarrow$ ": Trivial since $\langle \xi, p\xi \rangle + 2\operatorname{Re} \langle \xi, t\eta \rangle + \langle \eta, q\eta \rangle \geq 0$

(i): Directly follows from (ii)

(iii) By (ii) we have $|\langle z, t\eta \rangle|^2 \leq \langle z, pz \rangle \langle \eta, q\eta \rangle \leq \|p\| \|z\|^2 \langle \eta, q\eta \rangle$

Hence $\|t\eta\|^2 \leq \|p\| \langle \eta, q\eta \rangle$ (Fix η and take $\sup_{\|\eta\| \leq 1}$)

Since $\|t\eta\|^2 = \langle \eta, t^*t\eta \rangle$, $\langle (\|p\|q - t^*t)\eta, \eta \rangle \geq 0$

$$\Rightarrow t^*t \leq \|p\|q$$

Lemma 5.3 let $p: A \rightarrow B$ be a completely positive linear map between

C^* -algebras, and let (e_i) be an approximate unit for A . Then

$$(i) \|p\| = \sup_i \|p(e_i)\| \quad (ii) p^{(n)}(a^*) p^{(n)}(a) \leq \|p\| p^{(n)}(a^*a) \quad (a \in M_n(A))$$

Proof: (i) let $M = \sup_i \|p(e_i)\|$, clearly $M \leq \|p\|$. For $a \in A$, we have

$$\begin{pmatrix} e_i^2 & e_i a \\ a^* e_i & a^* a \end{pmatrix} = \begin{pmatrix} e_i & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} e_i & a \\ 0 & 0 \end{pmatrix} \geq 0$$

$$\Rightarrow \begin{pmatrix} p(e_i^2) & p(e_i a) \\ p(a^* e_i) & p(a^* a) \end{pmatrix} \geq 0, \text{ then by lemma 5.2 (iii) we have}$$

$$\underbrace{p(a^* e_i) p(e_i a)}_{\text{Here } p(a^*) = p(a) \text{ because } a = x_+ - x_- + i(y_+ - y_-) \text{ of positive elements and } p \text{ is positive and linear}} \leq \|p(e_i^2)\| p(a^* a) \leq \|p(e_i)\| p(a^* a)$$

Since $e_i^2 \leq e_i$, taking the limit we have $p(a^*) p(a) \leq M p(a^* a)$

$$\text{If } \|a\| \leq 1, \text{ then } \|p(a)\|^2 \leq M \|p(a^* a)\| \leq M \|p\| \Rightarrow \|p\|^2 \leq M \|p\| \Rightarrow \|p\| \leq M$$

$$\Rightarrow \|p\| = M$$

(ii): $n=1$ follows from the proof of (i). Now let $e_i^{(n)} = e_i \otimes 1 \in A \otimes M_n(\mathbb{C}) \cong M_n(A)$ then clearly $(e_i^{(n)})$ is an approximate unit for $M_n(A)$, and $\|p^{(n)}(e_i^{(n)})\| = \|p(e_i)\|$

Since $P^{(n)}$ completely positive, linear, by a similar proof of (i) we finish.

By (ii) we also have $\|P^{(n)}\| = \|P\|$ for all n . Hence P completely bounded.

Lemma 5.4 Let $P: A \rightarrow B$ be a completely positive mapping between C^* -algebras, and let $a_1, \dots, a_n \in A$. Then $(P(a_i^*)P(a_j)) \leq \|P\| (P(a_i^*a_j))$ in $M_n(B)$

proof: Apply lemma 5.3 (ii) to $\begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$

Now we consider completely positive maps in Hilbert C^* -modules.

Q: use completely positive map P to construct interior tensor product.

Let E Hilbert A -module, F Hilbert B -module and $P: A \rightarrow L(F)$ completely positive map.

Define a sesquilinear B -valued linear form on the right B -module

$E \otimes_{\text{alg}} F$ by $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, P(\langle x_1, x_2 \rangle) y_2 \rangle$ ($x_i \in E, y_i \in F$)

Same proof in prop 4.5

$\implies E \otimes_{\text{alg}} F$ semi-inner-product B -module. Then define $E \otimes_P F$ to be the Hilbert B -module obtained by completing $(E \otimes_{\text{alg}} F) / N_P$ where $N_P := \{ z \in E \otimes_{\text{alg}} F : \langle z, z \rangle = 0 \}$

Similar to Calculation 4.6, for $t \in L(E)$, if define $P_*(t) \cdot x \otimes y = tx \otimes y$, then $P_*(t)$ extends to $P_*(t) \in L(E \otimes_P F)$ (here $x \otimes y := x \otimes y + N_P$)

Furthermore, we still have $P_*: t \mapsto P_*(t)$ is a unital $*$ -homomorphism from $L(E)$ to $L(E \otimes_P F)$ which is strictly continuous on the unit ball.

Now we consider the case $E=A$, $P: A \rightarrow L(F)$ o.p., F Hilbert B -module.
First need to define non-degeneracy on P (follows from Prop 2.5), and define strictness for P .

Def (Prop 2.5(3)) $P: A \rightarrow L(F)$ is non-degenerate if $P(e_i) \rightarrow 1$ strictly in $L(F)$ for some approximate unit (e_i) of A .

Def We say P strict if $(P(e_i))$ is strict Cauchy in $L(F)$ for some approximate unit (e_i) in A .

clearly non-degenerate $\Rightarrow$ strict

Remark (i) If A unital, then P clearly strict, but P non-degenerate iff P unital.

(ii) If nonunital, then $C_0(0,1] \hookrightarrow C([0,1])$ clearly not strict.

(iii) Unit ball of $L(F)$ is complete for the strict topology. (proof omitted)

(iv) implies that $P: A \rightarrow L(F)$ is strict iff there is a positive element P in the ball of radius $\|P\|$ in $L(F)$ s.t. $P(e_i) \rightarrow P$ strictly.

Prop 5.5 Suppose that A, B are C^* -algebras, E, F are Hilbert B -modules, $\pi: A \rightarrow \mathcal{L}(E)$ is a non degenerate $*$ -homomorphism and $V \in \mathcal{L}(F, E)$. Then $P: A \rightarrow \mathcal{L}(F)$ defined by $P(a) = V^* \pi(a) V$ ($a \in A$) is a strict completely positive mapping.

proof: (i) completely positive: $\forall n \geq 1$, consider $P^{(n)}: M_n(A) \rightarrow M_n(\mathcal{L}(F))$

(clearly $M_n(\mathcal{L}(E)) \cong \mathcal{L}(E^n)$, $M_n(\mathcal{L}(F)) \cong \mathcal{L}(F^n)$, define $V_n = \text{diag}(V, \dots, V) \in \mathcal{L}(F^n, E^n)$

i.e. $V_n(\eta_1, \dots, \eta_n) = (V\eta_1, \dots, V\eta_n)$, then $V_n^* = \text{diag}(V^*, \dots, V^*)$

Notice that $\pi^{(n)}: M_n(A) \rightarrow \mathcal{L}(E^n)$ given by $\pi^{(n)}([a_{ij}]) = [\pi(a_{ij})]$

is a $*$ -homomorphism. Then for $X = [a_{ij}] \in M_n(A)$, we have

$$V_n^* \pi^{(n)}(X) V_n = [V^* \pi(a_{ij}) V] = [P(a_{ij})] = P^{(n)}(X)$$

Now for $X \geq 0$, set $X = y^* y$, then $P^{(n)}(X) = V_n^* \pi^{(n)}(y^* y) V_n = (\pi^{(n)}(y) V_n)^* (\pi^{(n)}(y) V_n)$
 $\Rightarrow P^{(n)}(X) \geq 0 \Rightarrow P$ completely positive.

(ii) strict: Since π nondegenerate, $\pi(e_i) \rightarrow 1$. Hence $P(e_i) \rightarrow V^* V$
 $\Rightarrow P$ strict.

Now we focus on the converse of prop 5.5.

For $P: A \rightarrow \mathcal{L}(F)$, $a_1, \dots, a_n \in A$, $y_1, \dots, y_n \in F$, we have

$$\left\| \sum_i P(a_i) y_i \right\|^2 = \left\| \sum_{i,j} \langle y_i, P(a_i^*) P(a_j) y_j \rangle \right\|$$

$$\leq \|P\| \left\| \sum_{i,j} \langle y_i, P(a_i^* a_j) y_j \rangle \right\| \quad (\text{by lemma 5.4})$$

$$= \|P\| \left\| \sum_i a_i \otimes y_i \right\|^2$$

Hence the map $a \otimes y \mapsto P(a)y$ extends to a bounded map $w: A \otimes_P F \rightarrow F$ s.t. $\|w\| \leq \|P\|^{\frac{1}{2}}$ and $\overline{w(A \otimes_P F)} = \overline{P(A)F}$

Now assume P is strict, with $P(e_i) \rightarrow P \in \mathcal{L}(F)$. Define $v_i: F \rightarrow A \otimes_P F$ by $v_i(y) = e_i \otimes y$. Then $\|v_i(y) - v_j(y)\|^2 = \|\langle y, P(e_i - e_j)^2 y \rangle\|$
 $\leq \|\langle y, P(e_i - e_j) y \rangle\|$

Since $P(e_i)$ Cauchy, $v_i(y)$ Cauchy in $A \otimes_P F$. Hence $v_i(y) \rightarrow$ some $v(y)$

For $a \in A$ and $y, z \in F$, we have $\langle a \otimes y, v(z) \rangle = \lim_i \langle a \otimes y, e_i \otimes z \rangle$

$$= \lim_i \langle y, P(a^* e_i) z \rangle = \langle y, P(a^*) z \rangle = \langle w(a \otimes y), z \rangle$$

$$\Rightarrow \langle x, v(z) \rangle = \langle wx, z \rangle \text{ for all } x \in A \otimes_P F \Rightarrow v = w^*$$

$$\Rightarrow v \in \mathcal{L}(F, A \otimes_P F) \text{ and } w \in \mathcal{L}(A \otimes_P F, F)$$

For $P_*: M(A) \rightarrow \mathcal{L}(A \otimes_P F)$ defined before ($P_*(c) \cdot a \otimes y = ca \otimes y$)

We claim that $P_*(A) \vee (F)$ is dense in $A \otimes_P F$. For $a \in A$ and $y \in F$

$$\text{we have } \|P_*(a) \cdot e_i \otimes y - a \otimes y\|^2 = \|(ae_i - a) \otimes y\|^2$$

$$= \|\langle y, P((1 - e_i) a^* a (1 - e_i) y) \rangle\| \xrightarrow{i \rightarrow \infty} 0 \Rightarrow P_*(a) \cdot y = a \otimes y$$

$\Rightarrow P_*(A) \vee F$ contains all finite sums $\Rightarrow$ dense

From now on, denote $F_p := A \otimes_p F$, π_p for the $*$ -homomorphism $P_*: M(A) \rightarrow L(F_p)$ and ν_p for the operator ν in $L(F, F_p)$. For $a \in A$ and $y \in F$, we have $\nu_p^* \pi_p(a) \nu_p y = \lim_i \nu_p^* \pi_p(a) \cdot e_i \otimes y = \lim_i \nu_p^* \cdot a e_i \otimes y$
 $= \lim_i P(a e_i) y = P(a) y \Rightarrow P(a) = \nu_p^* \pi_p(a) \nu_p$

Thm 5.6 let A, B be C^* -algebras, F be Hilbert B -module and

$P: A \rightarrow L(F)$ be a strict completely positive mapping.

(i) $\exists$ Hilbert B -module F_p , a $*$ -homomorphism $\pi_p: A \rightarrow L(F_p)$ and an element ν_p of $L(F, F_p)$, s.t. $P(a) = \nu_p^* \pi_p(a) \nu_p$ ($a \in A$)
 $\pi_p(A) \nu_p F$ is dense in F_p

(ii) If G is a Hilbert B -module, $\pi: A \rightarrow L(G)$ is a $*$ -homomorphism, $w \in L(F, G)$ and $P(a) = w^* \pi(a) w$ ($a \in A$), $\pi(A) w F$ is dense in G , then $\exists$ a unitary u in $L(F_p, G)$ s.t. $\pi(a) = u \pi_p(a) u^*$ ($a \in A$) and $w = u \nu_p$.

proof: (i) has been proved.

For (ii), let $a_1, \dots, a_n \in A$ and $y_1, \dots, y_n \in F$ then
 $|\sum_i \pi(a_i) w y_i|^2 = \sum_{i,j} \langle y_i, w^* \pi(a_i^* a_j) w y_j \rangle = \sum_{i,j} \langle y_i, P(a_i^* a_j) y_j \rangle$

$$= \sum_{i,j} \langle y_i, v_p^* \pi_p(a_i^* a_j) v_p y_j \rangle = |\sum_i \pi_p(a_i) v_p y_i|^2. \text{ Thus the}$$

map $u: \underbrace{\sum \pi_p(a_i) v_p y_i}_{F_p} \mapsto \underbrace{\sum \pi(a_i) \omega y_i}_{G_1}$ is (well-defined and) isometric.

Since $\pi_p(A) v_p F$ is dense in F_p and $\pi(A) \omega F$ dense in G_1 , the map u extends to a unitary map u from F_p onto G_1 .

$$\begin{aligned} \text{Since } u \pi_p(a) u^* \sum \pi(a_i) \omega y_i &= u \pi_p(a) \sum \pi_p(a_i) v_p y_i \quad (\text{since } \pi(a) = u \pi_p(a) u^*) \\ &= u \sum \pi_p(a a_i) v_p y_i \\ &= \sum \pi(a a_i) \omega y_i = \pi(a) \sum \pi(a_i) \omega y_i, \end{aligned}$$

we have $u \pi_p(a) u^* = \pi(a) \quad (a \in A)$.

By def of u we have $u(\pi_p(e_i) v_p y) = \pi(e_i) \omega y \quad (y \in F)$. Since π_p and π nondegenerate, both $\pi_p(e_i)$ and $\pi(e_i)$ converge strictly to the identity for $\lim_i (e_i)$. Hence $u v_p y = \omega y \Rightarrow u v_p = \omega$.

The unique triple (F_p, π_p, v_p) obtained by ρ will be called the KSGNS construction associated with ρ . If $F = B = \mathbb{C}$, then KSGNS reduces to classical GNS construction.

Coro 5.7 let A, B be C^* -algebras, let F be Hilbert B -module and $P : A \rightarrow \mathcal{L}(F)$ be completely positive. Then P is strict iff $\exists$ a completely positive map $\bar{P} : M(A) \rightarrow \mathcal{L}(F)$, strictly continuous on the unit ball, whose restriction to A is equal to P . Also, P is nondegenerate iff $\bar{P}$ is unital.

proof: If $\bar{P}$ exists, and (e_i) approximate unit for A , then $\bar{P}(e_i) \rightarrow \bar{P}(1)$ strictly, hence $P(e_i)$ converges, hence P is strict.

For the converse, assume P is strict. Then by Thm 5.6, $\exists$ $*$ -homomorphism $\pi_P : A \rightarrow \mathcal{L}(F_P)$ which is strictly continuous on the unit ball and therefore, by the results in chapter 2 (prop 2.5 (iii)) has an extension $\tilde{\pi}_P : M(A) \rightarrow \mathcal{L}(F_P)$. Define $\bar{P}(c) = v_P^* \tilde{\pi}_P(c) v_P$ ($c \in M(A)$). Then $\bar{P}$ satisfies the desired properties.

Since $P(e_i) \rightarrow \bar{P}(1)$, hence P nondegenerate iff $\bar{P}$ unital.

Recall prop 2.5 (three equivalent def of nondegeneracy)

(i) $P(A)F$ dense in F (ii) P is the restriction to A of a unital, completely positive map $\bar{P} : M(A) \rightarrow \mathcal{L}(F)$ which is strictly continuous on the unit ball
(iii) For (e_i) , $P(e_i) \rightarrow 1$ strictly.

By Cor 5.7 we have (ii) $\Leftrightarrow$ (iii). But (i) $\not\Rightarrow$ (iii): Consider $B = C([0,1])$
 $A = \{f \in B : f(0) = 0\}$ and define $P: A \rightarrow B$ by $P(f)(\lambda) = \frac{1}{2}(f(\lambda) + f(1-\lambda))$
 Initially P is completely positive and $P(A)B = B$ (take $f \in A: f(x) = x, 0 \leq x \leq 1$)
 Hence P satisfies (i), but not satisfies (iii) since $P(e_i)(0) \rightarrow \frac{1}{2}$

prop 5.8 Let A, B be C^* -algebras and let F be a Hilbert B -module.
 For a $*$ -homomorphism $\phi: A \rightarrow \mathcal{L}(F)$, the following conditions are equivalent:
 (i) $\phi(A)F$ is a complemented submodule of F ;
 (ii) ϕ is the restriction to A of a $*$ -homomorphism $\tilde{\phi}: M(A) \rightarrow \mathcal{L}(F)$ which is strictly continuous on the unit ball
 (iii) ϕ is strict.

If these conditions hold, then $\tilde{\phi}(1)$ is the projection from F onto $\overline{\phi(A)F}$.

proof: (prop 5.8 is an analogue of prop 2.5 for strictness)

since $\phi(A)F$ complemented,

(i) $\Rightarrow$ (ii): Let P be the projection from F onto $F_0 = \overline{\phi(A)F}$. Then $P \in \mathcal{L}(F, F_0)$ and P^* is the embedding of F_0 in F . For $a \in A$, define $\tilde{\phi}(a) = P\phi(a)P^*$.
 Then $\tilde{\phi}: A \rightarrow \mathcal{L}(F_0)$ is a nondegenerate $*$ -homomorphism (since P projection and ϕ $*$ -homomorphism, and $\phi(a)\xi \in F_0$ we have $\phi(e_i)\phi(a)\xi \rightarrow \phi(a)\xi$)

Then by extension we have $*$ -homomorphism $\tilde{\phi}: M(A) \rightarrow \mathcal{L}(F_0)$ which is strictly continuous on the unit ball. Define $\tilde{\phi}(c) = P^*\tilde{\phi}(c)P$ ($c \in M(A)$)

(ii) $\Rightarrow$ (iii): By corollary 5.7.

(iii) $\Rightarrow$ (i) : Assume ϕ strict, let $Q = \lim \phi(e_i) \in \mathcal{L}(F)$. Then for $\xi \in F$, we have $Q\xi = \lim \phi(e_i)\xi \Rightarrow Q\xi \in F_0 \Rightarrow QF \subseteq F_0 = \overline{\phi(A)F}$

Now for $\phi(a)\xi \in F_0$, we have $Q\phi(a)\xi = \lim \phi(e_i)\phi(a)\xi = \lim \phi(e_i a)\xi = \phi(a)\xi \Rightarrow Q\eta = \eta (\eta \in F_0) \Rightarrow F_0 \subseteq QF \Rightarrow F_0 = QF$

Since clearly $Q^2 = Q$ and $Q^* = Q$ (direct calculation), $Q = Q^* = Q^2$ and $QF = F_0 \Rightarrow F_0$ complemented (let $\xi = Q\xi + (1-Q)\xi$ and clearly $\langle Q\xi, (1-Q)\xi \rangle = 0$)

Moreover, $\bar{\phi}(1)$ is the projection from F onto $\overline{\phi(A)F}$ directly follows from the above proof.

For c.p map which is not strict, we can still write $\rho(a)$ as $V^* \pi(a) V$ where $\pi: A \rightarrow \mathcal{L}(E)$ and $V \in \mathcal{L}(F, E)$, but we may abandon the uniqueness and nondegeneracy.

Now we focus on a class of c.p. map.

Let A, B be C^* -algebras where B strictly closed in $M(A)$. A retraction from A to B is a positive linear map $\psi: A \rightarrow B$ s.t. the conditions
(i) $\psi(ab) = \psi(a)b$ (ii) $\psi(A)$ strictly dense in B

(iii) for (e_i) in A , $(\psi(e_i))$ converges strictly in $M(A)$ to a proj. $p \in B$.

If A unital, then a retraction is just the condition expectation (by (i) we get projection)

Lemma 5.9 If $\psi: A \rightarrow B \subseteq M(A)$ is a retraction then ψ is completely positive.
 proof: By a similar proof as lemma 4.3 (i), it suffices to prove that $(\psi(a_i^* a_j))_{i,j} \geq 0$ in $M_n(B)$ for all $a_1, \dots, a_n$ in A . Identifying $M_n(B)$ with $K(B^n)$, then this will hold if $\sum_{i,j=1}^n b_i^* \psi(a_i^* a_j) b_j \geq 0$ ($b_1, \dots, b_n \in B$) by lemma 4.1. Since ψ is positive and let $c = \sum_i a_i b_i$, then done.

prop 5.10 let A be C^* -algebra and B be a strictly closed C^* -subalgebra of $M(A)$. A linear map $\psi: A \rightarrow B$ is a retraction iff ψ is the restriction to A of an idempotent contraction $\bar{\psi}: M(A) \rightarrow B$ which is strictly continuous on the unit ball.
 proof: Assume $\psi: A \rightarrow B$ retraction. Since $\psi: A \rightarrow M(A)$ is completely positive by lemma 5.9 and strict, hence by Cor 5.7 ψ has an extension $\bar{\psi}: M(A) \rightarrow M(A)$ which is strictly continuous on the unit ball and is a contraction since $\|\bar{\psi}\| = \|\bar{\psi}(1)\| = \|\psi(1)\| = 1$. Since B strictly closed, the range of $\bar{\psi}$ is contained in B . Then by strict continuity we have $\bar{\psi}(cb) = \bar{\psi}(c)b$ ($c \in M(A), b \in B$). With $c=1$, then $\bar{\psi}(b) = p b p$ ($b \in B$) and $\bar{\psi}(p) = p$.

For $c \in M(A)$ with $0 \leq c \leq 1$ we have $0 \leq \bar{\psi}(c) \leq p$ and therefore $\bar{\psi}(c) = p \bar{\psi}(c) p$. Hence for $c \in M(A)$, we have $\bar{\psi}(c) = p \bar{\psi}(c) p$ ($c \in M(A)$). Then for $a \in A$, $\bar{\psi}(\psi(a)) = p \psi(a) p = \psi(a)$.

Hence $\bar{\psi}(b) = b$ for all $b \in B$ and so ψ is idempotent.

Conversely, if $\bar{\psi}: M(A) \rightarrow B$ is an idempotent contraction then it follows that $\bar{\psi}(cb) = \bar{\psi}(c)b$ ($c \in M(A), b \in B$)
 $\Rightarrow$ Done.

The localisation construction from Chapter 1 can be carried out for retractions.

let $\psi: A \rightarrow B$ retraction, then E defined in Chapter 1 can be made into a $M(A)$ -module (Calculation) and we can define $\langle, \rangle_\psi$ on E by

$$\langle x, y \rangle_\psi = \psi(\langle x, y \rangle) \quad (x, y \in E)$$

let $N_\psi := \{x \in E : \langle x, x \rangle_\psi = 0\}$ and complete E/N_ψ , then we get E_ψ and E_ψ is called the localisation of E by ψ .

For $t \in L(E)$, we define $\pi_\psi(t): x + N_\psi \mapsto tx + N_\psi$, then by extension we have unital $\ast$ -homomorphism $\pi_\psi: L(E) \rightarrow L(E_\psi)$.