Hilbert C^*-modules, Lecture 3

Chapter 4: Tensor Products

- Throughout, we consider only spatial C^*-algebra tensor products

- $A \odot B$ will denote the algebraic tensor product.

Here are some results that may (or may not) be useful for these notes: For C^*-algebras A & B, we have

(i) $A^* \odot B^*$ embeds in $(A \otimes B)^*$

(ii) for $r \in A \odot B$, the norm of r is

$$\|r\|^2 = \sup \left\{ \frac{(\rho \otimes \sigma)(t^* r^* r t)}{(\rho \otimes \sigma)(t^* t)} : \rho \in S(A), \sigma \in S(B), t \in A \odot B, (\rho \otimes \sigma)(t^* t) \neq 0 \right\}$$

where $S(A)$ & $S(B)$ are the set of states

(iii) If π & π' are faithful reps of A, B on $\mathcal{H}$ and $\mathcal{K}$, then $\pi \otimes \pi'$ are faithful reps of $A \otimes B$ on $\mathcal{H} \otimes \mathcal{K}$

(iv) The GNS representation of $\rho\otimes\sigma$, for states ρ and σ on A and B respectively, are unitarily equivalent to the representation $\pi_\rho\otimes\pi_\sigma$, where π_ρ, π_σ are the GNS rep of ρ & σ respectively.

Lemma 4.1 Let E be a Hilbert A-module & t be a bounded A-linear operator on E. TFAE
(i) t a positive
(ii) $\langle x, tx\rangle \geq 0 \quad \forall\, x\in E$

Proof: Recall the polarisation identity $\langle x,y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle x+(i)^k y, x+(-i)^k y\rangle$. So, to show t is self-adjoint, it suffices to show that $\forall\, x\in E$, $\langle x, tx\rangle = \langle tx, x\rangle$. If $t\geq 0$ in $\mathcal{L}(E)$, then $\langle t^{1/2}x, t^{1/2}x\rangle = |t^{1/2}x|^2 \geq 0$ in A. Conversely, by the comment above, if $\langle x, tx\rangle \geq 0 \ \forall\, x\in E$, then t is self-adjoint, because $\langle x, tx\rangle^* = \langle tx, x\rangle = \langle x, tx\rangle$. Write $t = r - s$ where $r, s \geq 0$ & $rs = 0$. For $x\in E$, we have $\langle sx, tsx\rangle \geq 0$ so $\langle x, s^3 x\rangle = -\langle x, s^3 x\rangle \geq 0$. But $s^3 \geq 0$, so we must have $\langle x, s^3 x\rangle = 0$. Therefore $s = 0$ & $t = r \geq 0$. ∎

- Recall the notation $E^n = \underbrace{E\oplus E\oplus \cdots E}_{n \text{ times}}$ and

$$\mathcal{L}(E^n) \simeq M_n(\mathcal{L}(E)), \quad \mathcal{K}(E^n) \simeq M_n(\mathcal{K}(E))$$

Lemma 4.2 Let E be a Hilbert A-module. If $x_1, \ldots, x_n \in E$ then $X \geq 0$ in $M_n(A)$, where X is the matrix with (i,j)-entries $\langle x_i, x_j\rangle$. Also, if $t\in\mathcal{L}(E)$ & W is the matrix with entries $\langle tx_i, tx_j\rangle$, then $W \leq \|t\|^2 X$.

Proof: Identify $M_n(A)$ with $\mathcal{K}(A^n)$. Then for all $\underline{a} = (a_1, \ldots, a_n)$ in A^n, we have

$$\langle a, Xa\rangle = \sum_{i,j} a_i^* \langle x_i, x_j\rangle a_j = \Big|\sum_i x_i a_i\Big|^2 \geq 0$$

Lemma 4.2, continued So, by Lemma 4.1, $X \geq 0$. Also,

$$\langle a, Wa\rangle = \sum_{i,j} a_i^* \langle tx_i, tx_j\rangle a_j = \left|\sum_i tx_i a_i\right|^2 \leq \|t\|^2 \left|\sum_i x_i a_i\right|^2$$

$$= \|t\|^2 \langle a, Xa\rangle$$

The inequality follows from prop 1.2.
Thus, $\langle a, (\|t\|^2 X - W)a\rangle \geq 0 \quad \forall\ a \in A^n$
So again, by lemma 4.1, $W \leq \|t\|^2 X$. ∎

Lemma 4.3 Let A, B be C^*-algebras & $n \in \mathbb{N}$. Suppose that the matrix $a = (a_{ij}) \geq 0$ in $M_n(A)$ & $b = (b_{ij}) \geq 0$ in $M_n(B)$. Then

(i) $(a_{ij} \otimes b_{ij}) \geq 0$ in $M_n(A \otimes B)$
(ii) $\sum_{i,j} a_{ij} \otimes b_{ij} \geq 0$ in $A \otimes B$

(iii) If $a \leq c = (c_{ij})$ in $M_n(A)$, and $b \leq d = (d_{ij}) \in M_n(B)$ then

$$(a_{ij} \otimes b_{ij}) \leq (c_{ij} \otimes d_{ij}) \quad \text{in } M_n(A \otimes B)$$

Proof: (i) Since $a \geq 0$, $a = s^* s$ for $s \in M_n(A)$. Thus $a_{ij} = \sum_k s_{ki}^* s_{kj}$. Similarly $b_{ij} = \sum_k t_{ki}^* t_{kj}$ in B. Thus, the matrix $(a_{ij} \otimes b_{ij})$ is the sum of matrices $\sum_m \sum_k (s_{ki}^* s_{kj} \otimes t_{mi}^* t_{mj})$. It suffices to show that each matrix in the sum is positive. Let $E = A \otimes B$ be the Hilbert $A \otimes B$ module, and $x_i := s_{ki} \otimes t_{mi}$. Consider $x_1, \ldots, x_n \in E^*$ and the matrix $X_{ij} = \langle x_i, x_j\rangle$. Then lemma 4.2 tells us $(X_{ij}) = (s_{ki}^* s_{kj} \otimes t_{mi}^* t_{mj})$ is a positive matrix, so we are done.

Lemma 4.3, continued

(ii) Let (u_α) and (v_β) be approximate units for A, B respectively. Also, let

$$e_{\alpha\beta} = (\underbrace{u_\alpha \otimes v_\beta, u_\alpha \otimes v_\beta, \dots u_\alpha \otimes v_\beta}_{n \text{ times}}) \in (A \otimes B)^n$$

It follows that from (i) that $\langle e_{\alpha\beta}, (a_{ij} \otimes b_{ij}) e_{\alpha\beta} \rangle \geq 0$ take the limit to get (ii).

(iii) By (i), the matrices $((c_{ij} - a_{ij}) \otimes b_{ij})$ and $(\sum c_{ij} \otimes (d_{ij} - b_{ij}))$ are both positive in $M_n(A \otimes B)$, Hence so is their sum.

Constructing Exterior tensor product of Hilbert C^*-modules

Throughout, A & B are C^*-algebras, and E is a Hilbert A-module & F a Hilbert B-module.

We define $E \otimes F$ as a Hilbert $(A \otimes B)$-module. Let $E \odot F$ be the algebraic tensor product as vector spaces. This is a right module over $A \odot B$ where $x \otimes y (a \otimes b) = xa \otimes yb$.

For $x_1, x_2 \in E$, $y_1, y_2 \in F$, define

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle$$

This is a sem-inner-product module over $A \odot B$. It is positive semi-definite because for $z = \sum x_i \otimes y_i \in E \odot F$, then

$$\langle z, z \rangle = \sum_{i,j} \langle x_i, x_j \rangle \otimes \langle y_i, y_j \rangle$$

By lemma 4.2, the matrices $(\langle x_i, x_j \rangle)$ & $(\langle y_i, y_j \rangle)$ are positive. So by lemma 4.3 (ii), $\langle z, z \rangle \geq 0$.

In fact, it is inner product, but too hard to prove. Will prove in Ch 6.

Taking the completion of this module as mentioned in chapter 1, gives us the exterior product $E \otimes F$.

Example: H a Hilbert space, A a C^*-algebra as a Hilbert A-module. Then $H \otimes A$ is a Hilbert $\mathbb{C} \otimes A \simeq A$-module.

Adjointable operators on $E \otimes F$

This section shows how $\mathcal{L}(E) \otimes \mathcal{L}(F)$ embeds in $\mathcal{L}(E \otimes F)$ for Hilbert C^*-modules E and F. Choose $s \in \mathcal{L}(E)$ and $t \in \mathcal{L}(F)$. Define $s \otimes t$ on $E \odot F$ by

$$s \otimes t\,(x \otimes y) = sx \otimes ty.$$

If $z = \sum_i x_i \otimes y_i \in E \odot F$, then

$$|(s \otimes t)z|^2 = \sum_{i,j} \langle sx_i \otimes ty_i,\ sx_j \otimes ty_j \rangle$$

$$= \sum_{i,j} \langle sx_i, sx_j \rangle \otimes \langle ty_i, ty_j \rangle$$

By the 2nd part of lemma 4.2, the matrix inequality $(\langle sx_i, sx_j \rangle) \leq \|s\|^2 (\langle x_i, x_j \rangle)$ holds & similarly, $(\langle ty_i, ty_j \rangle) \leq \|t\|^2 (\langle y_i, y_j \rangle)$. Then, by lemma 4.3 (iii), we have

$$\|s\|^2 \|t\|^2 (\langle x_i, x_j \rangle \otimes \langle y_i, y_j \rangle) - (\langle sx_i, sx_j \rangle \otimes \langle ty_i, ty_j \rangle) \geq 0$$

using the proof of lemma 4.3, we can turn this into a sum, to get

$$|(s \otimes t)z|^2 = \sum_{i,j} \langle sx_i, sx_j \rangle \otimes \langle ty_i, ty_j \rangle \leq \|s\|^2 \|t\|^2 \sum_{i,j} \langle x_i, x_j \rangle \otimes \langle y_i, y_j \rangle$$

Thus $\|(s \otimes t)z\|^2 \leq \|s\|^2 \|t\|^2 |z|^2$

This implies $\|(s \otimes t)z\|^* \leq \|s\| \|t\| \|z\|$. Therefore, we can extend $s \otimes t$ to be a bounded operator on all of $E \otimes F$.

It is not hard to show that in fact the extension belongs to $\mathcal{L}(E \otimes F)$, with its adjoint being the extension of $S^* \otimes t^*$. Therefore, we have a $*$-homomorphism $j: \mathcal{L}(E) \odot \mathcal{L}(F) \to \mathcal{L}(E \otimes F)$. With a little more work which we omit (see pg 36, 37), one can show j is bounded, and so extends to a $*$-homomorphism $j: \mathcal{L}(E) \otimes \mathcal{L}(F) \to \mathcal{L}(E \otimes F)$.

In fact, j is injective. This gives a canonical embedding of $M(A) \otimes M(B)$ into $M(A \otimes B)$ by treating them as Hilbert C^*-modules. j is only surjective if one of the modules is finite dimensional.

Example: Let H be an infinite-dimensional Hilbert space. By the above, $B(H) \otimes B(H)$ embeds into $B(H \otimes H)$, yet, the flip map $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ is not in $B(H) \otimes B(H)$.

Interior tensor product

Let E be a Hilbert A-module, F a Hilbert B-module. Suppose also we have a $*$-homomorphism $\phi: A \to \mathcal{L}(F)$. We want to construct $E \otimes_\phi F$ as a Hilbert B-module. To do this, we need to introduce some more concepts.

Lemma 4.4 Suppose E a Hilbert A-module, $x \in E$, and $0 < \alpha < 1$. Then there exists $w \in E$ s.t $x = w|x|^\alpha$.

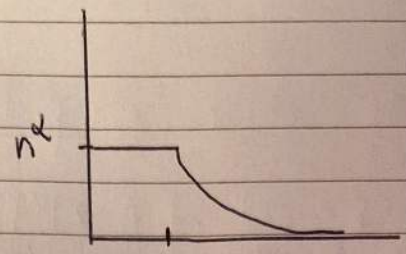
Proof: for any continuous function f on the spectrum of $|x|$, we have

$$\|x f(|x|)\| = \|f(|x|)^* \langle x, x \rangle f(|x|)\|^{1/2} = \||x| f(|x|)\|$$
$$= \sup\{|\lambda f(\lambda)| : \lambda \in sp(|x|)\}$$

For $n \geq 1$, define a function g_n by

$$g_n(\lambda) = \begin{cases} n^\alpha & \text{if } \lambda \leq 1/n \\ \lambda^{-\alpha} & \text{if } \lambda > 1/n \end{cases}$$

Lemma 4.4, cont. So the graph of g_n looks like



note that $(x g_n(|x|))$ is a cauchy sequence in E because

$$\sup_{\lambda \in sp(|x|)} (|\lambda(g_n(\lambda) - g_m(\lambda))| \leq \max(n^{\alpha-1}, m^{\alpha-1})$$

and $\alpha < 1$, so this goes to 0. Thus, $\exists$ limit w in E. Adjoining 1 to A if needed, & since $1x = 1x = x$ in E, we have

$$\|x g_n(|x|)|x|^\alpha - x\| = \|x(g_n(|x|)(|x|^\alpha - 1))\|$$

since $\sup_{\lambda \in sp(|x|)} \{|\lambda(g_n(\lambda)\lambda^\alpha - 1)| : \lambda \in sp(|x|)\} \to 0$

as $n \to \infty$, we have $w|x|^\alpha = x$. ∎

Definition: If $\varphi: A \to B$ a linear map between C^*-algebras, $\varphi^{(n)}$ is a map from $M_n(A)$ to $M_n(B)$ obtained by $\varphi^{(n)}((a_{ij})) = (\varphi(a_{ij}))$. φ is said to be **completely positive** if $\varphi^{(n)}$ positive map for all $n \in \mathbb{N}$.

If φ $*$-homomorphism, then it is completely positive.

Let E be an Hilbert A-module. E^n can be made into an $M_n(A)$-module as follows: for $a \in M_n(A)$, $x = (x_1, \ldots x_n) \in E^n$ let $xa = (x_1, \ldots x_n)\, a$ here multiplication is how you would normally multiply a $1 \times n$ matrix by an $n \times n$ matrix. So you consider elements of E^n as 'row vectors'. The inner product $\langle x, y \rangle$ is the matrix $(\langle x_i, y_j \rangle)$.

The fact that $\langle x,x\rangle \geq 0$ in this case follows from Lemma 4.2.

Constructing the Interior tensor product

Let E, f and $\phi: A \to \mathcal{L}(f)$ as before. f can be seen as a left A-module, with action given by $(a,y) \mapsto \phi(a)y$.

Take the algebraic vector space tensor product $E \odot f$, and quotient out by the subspace generated by

$$xa \otimes y - x \otimes \phi(a)y \qquad (x \in E,\ y \in f,\ a \in A) \quad (4.3)$$

B acts on this space by $(x \otimes y)b = x \otimes (yb)$. We denote this B-module by $E \odot_A f$.

Proposition 4.5 With A, B, E, f & ϕ as above, $E \odot_A F$ is an inner-product B-module with inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \phi(\langle x_1, x_2 \rangle) y_2 \rangle \qquad (4.4)$$

Proof: If $z = \sum_{i=1}^{n} x_i \otimes y_i \in E \odot f$, then

$$\langle z, z \rangle = \sum_{i,j} \langle y_i, \phi(\langle x_i, x_j \rangle)\, y_j \rangle$$

$$= \langle \underline{y}, \phi^{(n)}(X)\, \underline{y} \rangle$$

where $\underline{y} = (y_1, \dots y_n) \in f^n$, and $X \in M_n(A)$, where $X_{ij} = \langle x_i, x_j \rangle$. By lemma 4.2, $X \geq 0$. Since ϕ is a completely positive map, $\phi^{(n)}(X) \geq 0$, so $\langle z, z \rangle \geq 0$.

Thus, (4.4) turns $E \odot f$ into a semi inner-product B module. Let

$$N = \{ z \in E \odot f : \langle z, z \rangle = 0 \}$$

Proposition 4.5 continued We will show N is the same as the subspace generated by (4.3), so the semi inner-product factors through as an inner-product on $E \odot_A F$.

IF $z = \sum x a \otimes y - x \otimes \phi(a) y$, then

$$\langle z,z\rangle = \langle \sum xa\otimes y, \sum xa\otimes y\rangle - \langle \sum xa\otimes y, \sum x\otimes\phi(a)y\rangle - \langle \sum x\otimes\phi(a)y, \sum xa\otimes y\rangle + \langle \sum x\otimes\phi(a)y, \sum x\otimes\phi(a)y\rangle$$

$$= \langle y, \phi(a^*\langle x|x\rangle a)y\rangle - \langle y, \phi(a^*\langle x|x\rangle a)y\rangle - \langle y, \phi(a^*\langle x|a)y\rangle + \langle y, \phi(a^*\langle x|x\rangle a)y\rangle = 0$$

Conversely, suppose $z = \sum x_i \otimes y_i \in N$. Then using previous notation,

$$\langle z,z\rangle = \langle \underline{y}, \phi^{(n)}(X)\underline{y}\rangle = 0$$

for $y \in F^n$, $X \in M_n(A)$. Let $T = \phi^{(n)}(X)$. Then $T \geq 0$ in $M_n(\mathcal{L}(F)) \cong \mathcal{L}(F^n)$ and $T^{1/4} y = 0$.

Treat E^n as a $M_n(A)$-module as mentioned earlier. Let $\underline{x} = (x_1, \dots x_n)$ then little $\langle \underline{x}| = \sum X^{1/2}$ (big $X_{ij} = \langle \underline{x}_i, \underline{x}_j\rangle$). By lemma 4.4, $\exists\ \underline{w} \in E^n$ s.t. $\underline{w}\langle \underline{x}|^{1/2} = \underline{w} X^{1/4} = \underline{x}$. write c_{ij} for matrix elements of $X^{1/4}$.
Then $T^{1/4} = \phi^{(n)}(X^{1/4})$, so $(T^{1/4})_{ij} = \phi(c_{ij})$. So, we have

$$T^{1/4}\underline{y} = 0 \Rightarrow \forall\ j=1,\dots,n,\ \sum_i \phi(c_{ij})\, y_i = 0$$

and,

$$x_j = \sum w_i c_{ij}, \quad \text{since } \underline{w} X^{1/4} = \underline{x}.$$

So, $z = \sum_j x_j \otimes y_j$, using the formulas above, we can write

$$z = \sum_{ij} w_i c_{ij} \otimes y_j - w_i \otimes \phi(c_{ij}) y_j,$$ so z is of the form of (4.3). ∎

Completion of $E \odot_A F$ with this inner product is the Hilbert B-module denoted $E \otimes_\phi F$. we denote elements of $E \otimes_\phi F$ by $x \dot{\otimes} y$.

Example: $E = H \otimes A$, for a Hilbert space H, a C^*-algebra A. Write $H_A = H \otimes A$. Let F be a Hilbert B-module & suppose $\phi \in Mor(A, \mathcal{K}(F))$.

Then linear combinations of elements of the form $(\xi \otimes a) \dot{\otimes} y$ are dense in $H_A \otimes_\phi F$. By nondegeneracy of ϕ, sum of the form $\xi \otimes \phi(a) y$ are dense in $H \otimes F$. one can show that

$$\left\| \sum_i (\xi \otimes a_i) \dot{\otimes} y_i \right\| = \left\| \sum_i \xi_i \otimes \phi(a_i) y_i \right\| \qquad (4.5)$$

so the map $(\xi \otimes a) \dot{\otimes} y \mapsto \xi \otimes \phi(a) y$ extends to a unitary map from $H_A \otimes_\phi F$ to $H \otimes F$. So the two modules are unitarily equivalent.

Using this example when $F = B$, we get $H_A \otimes_\phi B \simeq H_B$.

Now suppose E is a Hilbert A-module, $t \in \mathcal{L}(E)$, F a Hilbert B-module & $\phi : A \to \mathcal{L}(F)$ $*$-hom. The map defined by $x \dot{\otimes} y \mapsto tx \dot{\otimes} y$ extends to a bounded map on $E \otimes_\phi F$. This map is denoted by $\phi_*(t) \in \mathcal{L}(E \otimes_\phi F)$

The map $\phi_* : t \mapsto \phi_*(t)$ is a unital nondegenerate $*$-hom from $\mathcal{L}(E)$ to $\mathcal{L}(E \otimes_\phi F)$. ϕ_* is injective iff ϕ is injective