

Recall: An ideal $I \trianglelefteq A$ (A C^* -algebra) is called essential if $\forall a \in A, aI = 0 \Rightarrow a = 0$.

The multiplier algebra $M(A)$ is the largest unital C^* -algebra containing A as an essential ideal.

There is also an explicit construction, see Murphy

Examples:

(i) A unital $\Rightarrow M(A) \cong A$

(ii) $A = C_0(X)$, X loc. cpt. T_2 space $\Rightarrow M(A) \cong C(\beta X)$ where βX is the Stone-Ćech compactification.

(iii) $A = C_0(X)$, X loc. cpt. T_2 space $\Rightarrow M(A) \cong C_b(X)$

Definition: Let C be a C^* -algebra and E a Hilbert C -module. A $*$ -hom. $\alpha: A \rightarrow \mathcal{L}(E)$ is called nondegenerate if $\alpha(A)E$ is dense in E

Convention of book: $\alpha(A)E$ means $\text{span} \{ \alpha(A)E \} = \{ \sum_{i=1}^n \alpha(a_i)x_i \}$

Some authors write $[\alpha(A)E] = E$ ($[\cdot]$ meaning closed linear span)

Proposition 1.2: A, B, C C^* -algebras s.t. $A \trianglelefteq B$ and E a Hilbert C -module. Suppose $\alpha: A \rightarrow \mathcal{L}(E)$ is nondegenerate $*$ -hom. Then α extends uniquely to a $*$ -hom. $\bar{\alpha}: B \rightarrow \mathcal{L}(E)$.

If α injective, A essential in B , then $\bar{\alpha}$ injective

Proof: Let (e_i) be an approximate identity for A . For $b \in B, a_1, \dots, a_n \in A$

Proof: Let (e_j) be an approximate identity for A . For $b \in B$, $a_1, \dots, a_n \in A$, $\xi_1, \dots, \xi_n \in E$

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha(a_i) \xi_i \right\| &= \lim_j \left\| \sum_{i=1}^n \alpha(b e_j a_i) \xi_i \right\| & \| \alpha(b e_j) \| &\leq \| b e_j \| \\ & & &= \| b \| \\ &= \lim_j \left\| \alpha(b e_j) \sum_{i=1}^n \alpha(a_i) \xi_i \right\| & b e_j &\in A \\ &\leq \| b \| \left\| \sum_{i=1}^n \alpha(a_i) \xi_i \right\| \end{aligned}$$

$\therefore$ The map $\sum_{i=1}^n \alpha(a_i) \xi_i \mapsto \sum_{i=1}^n \alpha(b e_j a_i) \xi_i$ is cb-

As α nondegenerate it extends to a bdd. map $\bar{\alpha}(b)$ on E . We see also $\bar{\alpha}(b^*)$ acts as an adjoint, so $\bar{\alpha}(b) \in \mathcal{K}(E)$. Easy to verify $\bar{\alpha}$ is a $*$ -hom. and unique.

α injective $\Rightarrow \ker \bar{\alpha} \trianglelefteq B$, $\ker \bar{\alpha} \cap A = 0$. A essential $\Rightarrow \ker \bar{\alpha} = 0$. $\square$

Recall: $E = A \Rightarrow \mathcal{K}(E) \cong A$.

Corollary: $M(A) \cong \mathcal{K}(A)$

Proof: Let $\alpha: A \rightarrow \mathcal{K}(A)$ be the canonical embedding, so

$$\alpha(ab^*) = \theta_{a,b}: c \mapsto a \langle b, c \rangle = ab^*c \Rightarrow \alpha(a): b \mapsto ab$$

(because $\text{span}\{A^*A\}$ dense in A , $\alpha \in \mathcal{K}(A)$)

A is an essential ideal in $M(A)$, and α injective so extends to an injective $\bar{\alpha}: M(A) \rightarrow \mathcal{K}(A)$

Now note under embedding α , A is an essential ideal in $\mathcal{K}(A)$ so the inclusion $\beta: A \rightarrow M(A)$ extends to $\bar{\beta}: \mathcal{K}(A) \rightarrow M(A)$

We may also consider the extension of $\alpha: A \rightarrow \mathcal{K}(A)$ to

We may also consider the extension of $\alpha: A \rightarrow \mathcal{L}(A)$ to $\tilde{\alpha}: \mathcal{L}(A) \rightarrow \mathcal{L}(A)$. As $\text{id}: \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ is also an extension, by uniqueness $\tilde{\alpha} = \text{id}$. $\tilde{\alpha} \circ \beta: \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ also extends α , so must be identity. $\therefore \tilde{\alpha}$ is surjective and thus an isomorphism. $\square$

Theorem 2-2: Let A be a C^* -algebra

- (i) $\mathcal{L}(A)$ is an essential extension of $\mathcal{K}(A)$ which is maximal (i.e. if $\mathcal{K}(A)$ is essential ideal of C , $\exists$ injective $*$ -hom. $C \rightarrow \mathcal{L}(A)$ s.t. identity map when restricted to $\mathcal{K}(A)$).
- (ii) If B is a maximal essential extension of A , there is a $*$ -iso. $B \rightarrow \mathcal{L}(A)$ its restriction onto A is the canonical map $A \rightarrow \mathcal{K}(A)$.

Proposition 2-3: Let A, C be C^* -algebras, E Hilbert C -module. Suppose $\alpha: A \rightarrow \mathcal{L}(E)$ is nondegenerate injective $*$ -hom. B the idealiser of $\alpha(A)$ in $\mathcal{L}(E)$ that is:

$$B = \{s \in \mathcal{L}(E) : s\alpha(A) \subseteq \alpha(A), \alpha(A)s \subseteq \alpha(A)\} \subseteq \mathcal{L}(E)$$

Then α extends to a $*$ -iso. $M(A) \xrightarrow{\sim} B$.

Proof: Straightforward to check $\alpha(A)$ is an essential ideal in B .

Suppose A is an essential ideal in D . Then by Prop. 2-1 $\alpha: A \rightarrow \mathcal{L}(E)$ extends to injective $*$ -hom. $\tilde{\alpha}: D \rightarrow \mathcal{L}(E)$. $A \trianglelefteq D \Rightarrow \tilde{\alpha}(D) \subseteq B$

So D embeds in B , thus B is maximal.

$\therefore$ By Theorem 2-2 (ii) it extends to $*$ -iso. $M(A) \xrightarrow{\sim} B$.

Theorem 2-4: E Hilbert A -module $\Rightarrow \mathcal{L}(E) \cong M(\mathcal{K}(E))$

Proof: $i: \mathcal{K}(E) \hookrightarrow \mathcal{L}(E)$ is nondegenerate and the idealiser of $\mathcal{K}(E)$ is

Proof: $i: \mathcal{K}(E) \hookrightarrow \mathcal{L}(E)$ is nondegenerate, and the idealiser of $\mathcal{K}(E)$ is $\mathcal{L}(E)$. So Prop. 2-3 $\Rightarrow i$ extends to $*$ -iso. $M(\mathcal{K}(E)) \cong \mathcal{L}(E)$.

Example: $M(\mathcal{K}(H)) \cong \mathcal{L}(H)$ for H Hilbert space (we view as a Hilbert $\mathbb{C}$ -module).

Proposition 2.5: A, B C^* -algebras, E Hilbert B -module. $*$ -hom. $\alpha: A \rightarrow \mathcal{L}(E)$
TFAE

(i) α is nondegenerate (recall $[\alpha(A)E] = E$)

(ii) α is the restriction to A of a unital $*$ -hom. $\bar{\alpha}: M(A) \rightarrow \mathcal{L}(E)$ which is strictly cb. on unit ball.

(iii) For an approx. identity (e_i) of A $\alpha(e_i) \rightarrow 1$ strictly

Proof:

(i) $\Rightarrow$ (ii): Suppose a net $x_i \rightarrow x$ strictly in $\bar{B}^{M(A)}$ so $\|(x_i - x)_a\| \xrightarrow{i} 0$
 $\forall a \in A$. For $a \in A, \xi \in E$

$$\|(\bar{\alpha}(x_i) - \bar{\alpha}(x))\alpha(a)\xi\| = \|\alpha((x_i - x)_a)\xi\| \xrightarrow{i} 0$$

for $\bar{\alpha}: M(A) \rightarrow \mathcal{L}(E)$ extension

α nondegenerate, $\|x_i - x\|$ bdd. $\Rightarrow \|(\bar{\alpha}(x_i) - \bar{\alpha}(x))\xi\| \xrightarrow{i} 0 \quad \forall \xi \in E$.

$\therefore \bar{\alpha}(x_i) \rightarrow \bar{\alpha}(x)$ strictly. $\bar{\alpha}$ is clearly unital.

(ii) $\Rightarrow$ (iii) Let (e_i) approx. unit $e_i \xrightarrow{i} 1$ strictly in $M(A)$
(follows as Prop 1-3 $\Rightarrow \bar{B}^A$ is self-adj. 1. $\bar{\alpha}(M(A))$)

... approx. unit $e_i \xrightarrow{i} 1$ strictly in $M(A)$
 (follows as Prop 1.3 $\Rightarrow \bar{B}^A$ is strictly dense in $\bar{B}^{M(A)}$)

$\therefore \bar{\alpha}(e_i) \xrightarrow{i} 1$ strictly

(iii) $\Rightarrow$ (i) $\alpha(e_i) \rightarrow 1$ strictly $\Rightarrow \alpha(e_i) \xi \rightarrow \xi \quad \forall \xi \in E \Rightarrow \alpha(A)E$ dense in E . $\square$

For A, B C^* -algebras, recall $\alpha: A \rightarrow M(B)$ nondegenerate $*$ -hom.
 (we can define nongenerate using isomorphism $M(B) \cong \mathcal{K}(B)$) extends to
 nondegenerate $\bar{\alpha}: M(A) \rightarrow M(B)$. Define a morphism $\text{Mor}(A, B)$
 $= \{ \alpha: A \rightarrow M(B) \text{ nondegenerate} \}$.

Proposition: A, B, C C^* -algebras, $\alpha \in \text{Mor}(A, B)$, $\beta \in \text{Mor}(B, C)$ then
 $\bar{\beta}\alpha \in \text{Mor}(A, C)$

Proof: $\bar{\beta}\alpha$ extends to $\bar{\beta}\bar{\alpha}: M(A) \rightarrow M(C)$. By Proposition 2.5 (i) $\Rightarrow$ (ii),
 $\bar{\beta}, \bar{\alpha}$ are strictly cb. on the unit ball, so it is clear that
 $\bar{\beta}\bar{\alpha}$ also must be and thus ((ii) $\Rightarrow$ (i)) is nondegenerate. $\square$

So the above really is a notion of morphism if we interpret
 $\beta \circ \alpha = \bar{\beta}\alpha$.

§3. Projections and Unitaries

Definition: Let E be a Hilbert A -module, and F a closed submodule of
 E . Then F is complemented if $E = F \oplus F^\perp$

We can define projections $p: E \rightarrow F$ s.t. $p \in \mathcal{K}(E)$.

Conversely $p \in \mathcal{L}(E)$ projection $\Rightarrow \text{ran } p$ is complemented as
 $(\text{ran } p)^\perp = \text{ran}(1-p) = \ker p, \quad E = \text{ran } p \oplus \text{ran}(1-p).$

Definition: Let E be a Hilbert A -module. We say F closed submodule of E is topologically complemented if $\exists$ closed submodule G s.t. $F \cap G = \{0\}, \quad E = F + G.$

Δ Topologically complemented $\nRightarrow$ orthogonally complemented.

Example: $A = C([0, 1]), \quad J = \{f \in A : f(0) = 0\}, \quad \text{Let } E = A \oplus J,$
 $F = \{(f, f) : f \in J\}.$

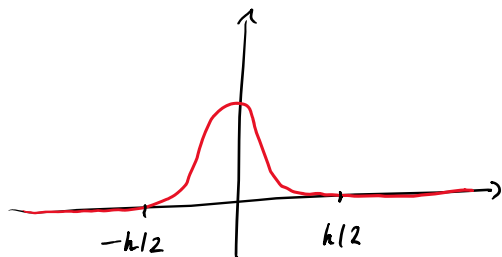
Then $F^\perp = \{(g, -g) : g \in J\} \Rightarrow F \oplus F^\perp = J \oplus J \neq E$

But $G = \{(f, 0) : f \in A\}$ is a topological complement.

"Projections" to topological complements are not necessarily adjointable

Lemma 3.1: Let E Hilbert module, $t^* = t \in \mathcal{L}(E)$ so that $\exists k > 0$ with $\|tx\| \geq k\|x\| \quad \forall x \in E.$ Then t is invertible.

Proof: We show $0 \notin \sigma(t)$. Let $f \in C(\mathbb{R})$ be s.t. $f(0) = 1 = \|f\|,$
 $f(\lambda) = 0 \quad \forall |\lambda| \geq k/2$



Using functional calculus on $C^*(t) \subseteq \mathcal{L}(E),$ let $s = f(t).$ Then
 $\|s\| = 1 \quad \|ts\| \leq k/2 \quad \text{Choose } x \in E \text{ s.t. } \|x\| = 1 \text{ with } \|s_2 x\| > 1/2 \quad \text{Then}$

Using functional calculus on $\mathcal{L}(E)$, let $s = f(t)$. Then $\|s\| = 1$, $\|ts\| \leq k/2$. Choose $x \in E$ s.t. $\|x\| = 1$ with $\|sx\| > 1/2$. Then $\|tsx\| \leq k/2 < k\|sx\|$. $\therefore 0 \notin \sigma(t)$. $\square$

Theorem 3.2: E, F Hilbert A -modules, $t \in \mathcal{L}(E, F)$ s.t. $\text{ran } t$ closed. Then:

- (i) $\ker t$ is complemented in E
- (ii) $\text{ran } t$ is complemented in F
- (iii) $t^* \in \mathcal{L}(F, E)$ has closed range.

Proof: Omitted.

Corollary 3.3: q idempotent in $\mathcal{L}(E) \Rightarrow \text{ran } q$ complemented.

Proof: Range of bdd. idempotent map is closed. $\square$

Definition: An operator $u \in \mathcal{L}(E, F)$ is unitary if $u^*u = 1_E$, $uu^* = 1_F$.

If there exists a unitary element, we say E, F are unitarily equivalent and write $E \approx F$.

Note that $u \in \mathcal{L}(E, F)$ unitary $\Rightarrow u$ is a surjective isometry.

Lemma 3.4: Let A be a C^* -algebra, $a, b \in A^+$ s.t. $\forall c \in A$ $\|ac\| = \|bc\|$. Then $a = b$.

Proof: WLOG, $\|a\|, \|b\| \leq 1$. Suffices to show $a^2 = b^2$. Suppose $a^2 - b^2 \neq 0$. $\Rightarrow \sigma(a^2 - b^2) \neq \{0\}$. Let $\delta = \frac{1}{2} \sup \sigma(a^2 - b^2) > 0$ (exchanging a, b if needed). Let $f: \sigma(a^2 - b^2) \rightarrow \mathbb{R}$ s.t. satisfying $0 \leq f \leq 1$, $f(\lambda) = 0$ if $\lambda \leq \delta$, $f(2\delta) = 1$.

$$c = f(a^2 - b^2) \in A.$$

$$\Rightarrow \|c(a^2 - b^2)c\| = 2\delta.$$

$$f(\lambda) \neq 0 \Rightarrow \lambda > \delta, \text{ so } c(a^2 - b^2)c \geq \delta c^2$$

Let ρ be a state of A s.t. $\rho(cb^2c) = \|cb^2c\|$. Since $\|b\| \leq 1$, we have

$$\rho(cb^2c) \leq \rho(c^2) \Rightarrow \rho(ca^2c) \geq \rho(cb^2c + \delta c^2) \geq (1 + \delta)\rho(cb^2c)$$

$$\Rightarrow \|ca^2c\| \geq (1 + \delta)\|cb^2c\|.$$

$$\therefore cb^2c \neq 0 \Rightarrow \|ca^2c\| > \|cb^2c\|.$$

$$\text{If } cb^2c = 0, \text{ then } ca^2c \neq 0 \Rightarrow \|ca^2c\| > \|cb^2c\|$$

□

$$\therefore \|ac\| > \|bc\| \quad \text{if}$$

Theorem 3.5: Let A C^* -algebra, E, F Hilbert A -modules and $u: E \rightarrow F$ a linear map. Then u is an isometric, surjective A -linear map iff u is unitary in $\mathcal{L}(E, F)$.

Proof: ($\Rightarrow$) Let $x \in E, a \in A$. Then

$$\begin{aligned} \| |ux| a \| &= \| a^* \langle ux, ux \rangle a \|^{1/2} \\ &= \| \langle u(xa), u(xa) \rangle \|^{1/2} = \| u(xa) \| = \| xa \| \\ &= \| a^* \langle x, x \rangle a \|^{1/2} = \| |x| a \| \end{aligned}$$

$$\text{Lemma 3.4} \Rightarrow |ux| = |x|$$

$$\Rightarrow \langle ux, ux \rangle = \langle x, x \rangle$$

$$\text{By polarisation } \langle ux, uy \rangle = \langle x, y \rangle \quad \forall x, y \in E.$$

u invertible, so we can see $u^* = u^{-1}$.

($\Leftarrow$) Easy. □

Proposition 3.6: Let E, F be Hilbert A -modules, $w: E \rightarrow F$ linear. Then w is an isometric, A -linear map with complemented range iff $w \in \mathcal{L}(E, F)$ and $w^*w = I_E$.

Proof: ($\Rightarrow$) Via the same method as Thm. 3.5 we can show

$$\langle wx, wy \rangle = \langle x, y \rangle \quad \forall x, y \in E.$$

As $\text{ran } w$ is complemented $\exists$ projection $p \in \mathcal{L}(F)$, $p(F) = \text{ran } w$. So we can then check $w^{-1}p = w^*$.

($\Leftarrow$) Check ww^* is a projection, and $\ker(ww^*)$ is a complement for $\text{ran } w$. □

Example: $H \otimes A \approx \bigoplus_i A_i \quad (A_i = A)$

Let $\{\varepsilon_i\}_{i \in I}$ be an o.n. basis. Then $u: \varepsilon_i \otimes a \mapsto (a\delta_{ij})_{j \in I}$

Example: $H \otimes C_0(X) \approx C_0(X, H)$ (as Hilbert $C_0(X)$ modules)

$$v: H \otimes C_0(X) \rightarrow C_0(X, H)$$

$$\xi \otimes f \mapsto (x \mapsto f(x)\xi)$$

Proposition 3.7: $t \in \mathcal{L}(E, F)$, $\overline{t^*F} = \overline{t^*tE}$

Proof: Let $J = \overline{\{t^*tu : u \in \mathcal{L}(E)\}}$, $K = \overline{\{t^*v : v \in \mathcal{L}(E, F)\}}$

(norm closure) be right ideals of $\mathcal{L}(E)$.

$$\overline{JE} = \overline{t^*tE} \quad \text{is clear}$$

For $x, y \in E$, $z \in F$, $\theta_{z,x} \in \mathcal{L}(E, F)$, $\theta_{z,x}(y) = z \langle x, y \rangle$

$$\Rightarrow F \langle E, E \rangle \subseteq \mathcal{L}(E, F)E$$

Let (e_i) be approx. unit for $\overline{\langle E, E \rangle}$ and let $y \in F$. Then

$$t^*y = \lim_i (t^*y)e_i$$

$$t^*(ye_i) \in t^*(\overline{F \langle E, E \rangle}) \subseteq t^*(\overline{\mathcal{L}(E, F)E}) \subseteq \overline{KE}$$

$$\Rightarrow t^*F \subseteq \overline{KE}$$

Conversely, $KE \subseteq \overline{t^*F}$

$$\therefore \overline{t^*F} = \overline{KE}$$

So it suffices to show $J = K$.

Let ρ be a state of $\mathcal{L}(E)$ that vanishes on J . Then $\rho(t^*t) = 0$.

$(u, v) \mapsto \rho(u^*v)$ satisfies CS

$$\therefore |\rho(t^*u)|^2 \leq \rho(t^*t)\rho(u^*u) = 0$$

$\Rightarrow \rho$ vanishes on K .

□

$$\therefore J = K$$

Proposition 3.8: E, F Hilbert A modules and $\exists t \in \mathcal{L}(E, F)$ s.t. t, t^* have dense range. Then $E \approx F$.

Proof: By Prop. 3.7 t^*t has dense range, so $|t| = (t^*t)^{\frac{1}{2}}$ also does. Define $u: \text{ran } t \rightarrow \text{ran } |t|$

$$tx \mapsto |t|x$$

For $x, y \in E$, $\langle utx, utx \rangle = \langle x, |t|^2 y \rangle = \langle tx, ty \rangle$

$\Rightarrow u$ isometric, and by the dense ranges extends to a unitary map $u: E \rightarrow F$. $\square$

An element $c \in \mathcal{L}(E, F)$ is called a partial isometry (from E_0 to F_0) if $F_0 = \text{ran } c$ is complemented in F , and $\exists$ complemented submodule E_0 of E s.t. c is isometric from E_0 to F_0 and $c(E_0^\perp) = \{0\}$.

One can now verify TFAE

- (i) c is a partial isometry
- (ii) c^*c is a projection in $\mathcal{L}(E)$
- (iii) cc^* is a projection in $\mathcal{L}(F)$
- (iv) $cc^*c = c$, (v) $c^*cc^* = c^*$.

Finally we can see that if $t \in \mathcal{L}(E, F)$ is s.t. $\text{ran } t, \text{ran } (t^*)$ are complemented. So t has polar decomposition $t = c|t|$ $c \in \mathcal{L}(E, F)$ a partial isometry.