

Notes on ‘Hilbert C* Modules’ by E.C. Lance

Noam Zeise

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Week 1

1.1. Motivation

Let A be a commutative unital C*-algebra, then via the Gelfand representation we know there is a compact space X such that

$$A \cong C(X) = \{f : X \rightarrow \mathbb{C} \mid f \text{ continuous}\},$$

via the Gelfand representation. If the space X were a manifold, we could consider a vector space H , and assign to each point $t \in X$ a closed subspace $H_t \subseteq H$. Then the set of functions

$$E = \{\xi : X \rightarrow H \mid \xi \text{ continuous, with } \xi(t) \in H_t, \forall t \in X\}$$

is a vector bundle. This space has a $C(X)$ -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow C(X)$ given by

$$\langle \xi, \eta \rangle \mapsto (t \mapsto \langle \xi(t), \eta(t) \rangle_H).$$

E is made a $C(X)$ -module by defining for $\xi \in E$ and $f \in C(X)$ the product ξf given by $t \mapsto \xi(t)f(t)$, an element of E .

This can be extended to arbitrary compact Hausdorff spaces X , by requiring H_t to be a closed subspace, a prototype of a Hilbert $C(X)$ -module.

1.2. Basic Definitions

For a C*-algebra A , a **right A-module** is a vector space E which is acted on by A via right multiplication $E \times A \rightarrow E$ such that for all $x, y \in E$ and $a, b \in A$ we have

- (i) $(x + y)a = xa + ya$,
- (ii) $x(a + b) = xa + xb$,
- (iii) $x(ab) = (xa)b$.

Definition 1.1 (inner-product C*-module) Let A be a C*-algebra. A right A -module E is called an *inner-product A-module* if for all $x \in E$, $\lambda \in \mathbb{C}$, $a \in A$ we have

$$\lambda(xa) = (\lambda x)a = x(\lambda a),$$

and there is a map $E \times E \rightarrow A$, $(x, y) \mapsto \langle x, y \rangle$ such that for all $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$, and $a \in A$ the following hold

- (i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$,
- (ii) $\langle x, ya \rangle = \langle x, y \rangle a$,
- (iii) $\langle y, x \rangle = \langle x, y \rangle^*$,
- (iv) $\langle x, x \rangle \geq 0$,
- (v) $\langle x, x \rangle = 0$ implies that $x = 0$.

When the final condition is not satisfied, we call E a *semi-inner-product A-module*

Warning – We have defined the inner product to be linear in the second variable, and conjugate linear in the first.

With a simple application of the inner-product rules we get the following identity. Let $x, y \in E$, $a \in A$, then we have

$$\langle xa, y \rangle = \langle y, xa \rangle^* = (\langle y, x \rangle a)^* = a^* \langle x, y \rangle.$$

Proposition 1.2 Let E be a semi-inner-product A -module, for $x, y \in E$ we have

$$\langle y, x \rangle \langle x, y \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle.$$

The proof makes use of a standard result for C*-algebras, that for $c \geq 0$ we have

$$a^* c a \leq \|c\| a^* a.$$

Proof. Let $a \in A$, then

$$\begin{aligned} 0 &\leq \langle xa - y, xa - y \rangle \\ &= \langle xa, xa \rangle - \langle xa, y \rangle - \langle y, xa \rangle + \langle y, y \rangle \\ &= a^* \langle x, x \rangle a - a^* \langle x, y \rangle - \langle y, x \rangle a + \langle y, y \rangle \\ &= \|\langle x, x \rangle\| a^* a - a^* \langle x, y \rangle - \langle y, x \rangle a + \langle y, y \rangle. \end{aligned}$$

If we set $a = \langle x, y \rangle$, and assume $\|\langle x, x \rangle\| = 1$, we reduce to

$$0 \leq \langle y, x \rangle \langle x, y \rangle - \langle y, x \rangle \langle x, y \rangle - \langle y, x \rangle \langle x, y \rangle + \langle y, y \rangle = \langle y, y \rangle - \langle y, x \rangle \langle x, y \rangle.$$

Thus $\langle y, x \rangle \langle x, y \rangle \leq \langle y, y \rangle$.

For an arbitrary x we let $\lambda = \frac{1}{\sqrt{\langle x, x \rangle}}$, then $\|\langle \lambda x, \lambda x \rangle\| = 1$, so

$$\langle y, \lambda x \rangle \langle \lambda x, y \rangle \leq \langle y, y \rangle = \|\langle \lambda x, \lambda x \rangle\| \langle y, y \rangle.$$

Thus by cancelling a λ^2 term on each side we recover the desired inequality. $\square$

1.3. Submodules, Norms, and Absolute Values

For E a semi-inner-product A module, take the kernel

$$N = \{x \in E \mid \langle x, x \rangle = 0\}.$$

Then N is a *sub- A -module*. To see this, take $x, y \in E$, then $\langle y, x \rangle \langle x, y \rangle = \langle x, y \rangle^* \langle x, y \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle = 0$. Thus $\langle x + y, x + y \rangle = 0$. We then define an inner product A -module E/N by

$$\langle x + N, y + N \rangle = \langle x, y \rangle$$

We can define a norm on an inner-product A -module E by $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ for $x \in E$ (When E is a semi-inner-product, this is a semi-norm).

For each a in a C^* -algebra A , the element a^*a is normal. Thus we can define an element $|a| = (a^*a)^{\frac{1}{2}}$. Similarly for $x \in E$ we define a positive element $|x| = \langle x, x \rangle^{\frac{1}{2}}$. Then for $x, y \in E$, $a \in A$, the following natural inequalities hold:

- $|\langle x, y \rangle| \leq \|x\| \|y\|$ (by Proposition 1.2)
- $\|xa\| \leq \|x\| \|a\|$ (as $\|xa\|^2 = \|\langle xa, xa \rangle\| = \|a^* \langle x, x \rangle a\| \leq \|x\|^2 \|a^*a\|$)

1.4. Completions

Definition 1.3 (Hilbert C^* -Module) An inner-product A -module E that is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ is called a *Hilbert A -module*.

Given an inner product A -module E_0 , there exists a unique Hilbert A -module completion E by defining for all sequences $(x_n), (y_n)$ in E_0 ,

$$\langle x, y \rangle := \lim_{n \rightarrow \infty} \langle x_n, y_n \rangle,$$

where we use completeness of the C^* -algebra A .

Oppositely, given a pre- C^* -algebra A_0 (that is, not complete) we can loosen Definition 1.1 to allow for an inner-product A_0 -module E . All the same results hold, except the completion of E just described. If E is complete (A Hilbert A_0 -module), then by Proposition 1.2 we can extend the action of A_0 on E to a C^* -algebra A acting on E , so that it becomes a Hilbert A -module.

Now, for A_0 a pre- C^* -algebra, and E_0 an inner-product A_0 -module, we can form a completion to a Hilbert C^* -module. E_0 is a normed vector space, so we construct the completion to a Banach space E . We similarly complete A_0 to the C^* -algebra A . We know $\|\langle x, y \rangle_{A_0}\| \leq \|x\| \|y\|$, so we can extend the inner product to a continuous map $E \times E \rightarrow A$. Similarly, the action by A_0 is bounded $\|xa\| \leq \|x\| \|a\|$ so it extends to a right action $E \times A \rightarrow A$. Thus E becomes a Hilbert A -module. It remains to check that the properties of an inner-product A -module are satisfied.

1.5. Approximate Units and Ideals

Let E be a Hilbert A -module. For an approximate unit $(e_i)_{i \in I}$ of A we have for $x \in E$

$$\langle x - xe_i, x - xe_i \rangle = \langle x, x \rangle - e_i \langle x, x \rangle - \langle x, x \rangle e_i + e_i \langle x, x \rangle e_i$$

which converges to 0. Thus any element in E can be approximated by elements in EA , so EA is dense in E . Thus for A unital we have $x1 = x$. Given A^+ the unitisation of A , E becomes an A^+ -module by adding the relation $x(0, 1) = x$.

Let

$$\langle E, E \rangle = \text{span} \{ \langle x, y \rangle \mid x, y \in E \}$$

and take the closure $B = \overline{\langle E, E \rangle}$. Then B is a two-sided ideal of A . We show one side. Let $a \in A$, $b \in B$, then there are convergent nets (λ_i) in $\mathbb{R}$, $(x_i), (y_i)$ in E , such that

$$c = \lim_{\{i\}} \lambda_i \langle x_i, y_i \rangle$$

then

$$ca = \lim_i \lambda_i \langle x_i, y_i \rangle a = \lim_i \langle \overline{\lambda_i} x_i, y_i a \rangle = \langle \lim_i \overline{\lambda_i} x_i, \lim_i y_i a \rangle$$

but EA is dense in E , so $\lim_i y_i a = z \in E$, thus $ca = \lim_i \lambda_i \langle x_i, z \rangle \in B$, as required.

We also have that $E\langle E, E \rangle$ is dense in E , because if (u_i) is an approximate unit for B , then $xu_i \rightarrow x$ for each $x \in E$.

1.6. Examples

- Let A be a C^* -algebra, then A is an A -module with $\langle a, b \rangle = a^*b$.

- For $\{E_i\}$ a finite set of Hilbert A -modules, $\oplus E_i$ is a Hilbert A -module with $\langle x, y \rangle = \sum \langle x_i, y_i \rangle$. We define E^n to be the direct sum of the same A -module n times.
- For $\{E_i\}_{i \in I}$ a countable set of Hilbert A -modules, we define $E = \oplus E_i$ to be only the elements $x = (x_i)$ where $\sum \langle x_i, x_i \rangle$ converges in A . Then we define $\langle x, y \rangle = \sum \langle x_i, y_i \rangle$. For a finite set $J \subset I$, form the finite sum $\hat{E} = \oplus E_j$, then for $x = (x_i), y = y_i \in E$ restrict these to $\hat{x}, \hat{y}$ in $\hat{E}$. Then by Proposition 1.2 we have $\langle \hat{y}, \hat{x} \rangle \langle \hat{x}, \hat{y} \rangle \leq \|\hat{x}, \hat{x}\| \|\hat{y}, \hat{y}\|$, thus

$$\left\| \sum_{j \in J} \langle x_j, y_j \rangle \right\|^2 \leq \left\| \sum_{j \in J} \langle x_i, x_i \rangle \right\| \left\| \sum_{j \in J} \langle y_i, y_i \rangle \right\|$$

but each term on the right converges over the whole of I , so the inner product is well defined.

- For H a Hilbert space, the algebraic tensor product $H \otimes_a A$ has an inner product

$$\langle \xi \otimes a, \eta \otimes b \rangle = \langle \xi, \eta \rangle a^* b$$

defined on simple tensors, taking values in A . It needs to be shown that the inner product satisfies $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ iff $x = 0$. Then the completion $H \otimes A$ can be identified with the Hilbert A -module $\oplus A_i$

- Take A unital, $B \subset A$ a unital C*-subalgebra, and $\varphi : A \rightarrow B$ a linear norm-reducing idempotent (called a *conditional expectation*). φ is positive and satisfies

$$\varphi(bac) = b\varphi(a)c$$

φ is called *faithful* if for $a \in A$ positive we have $\varphi(a) = 0 \implies a = 0$. We construct the *localisation*. Let E be a Hilbert A -module, then E is a semi-inner-product B -module by $\langle x, y \rangle_B = \varphi(\langle x, y \rangle_A)$. For φ faithful, this is an inner-product.

1.7. Similarities to Hilbert Spaces

The goal of a Hilbert C*-module is to generalise the idea of a Hilbert space, so we expect analogues to certain properties.

For E a Hilbert A -module, like with Hilbert spaces, we have

$$\|x\| = \sup\{\|\langle x, y \rangle\| \mid y \in E, \|y\| \leq 1\}$$

for each $x \in E$.

We also define the orthogonal complement of a closed submodule $F \subset E$ as

$$F^\perp = \{y \in E \mid \langle x, y \rangle = 0, \forall x \in F\},$$

which is itself a closed submodule. But in general we do not have $E = F \oplus F^\perp$, as the following example shows.

Example 1.4 Let $A = C(X)$, X compact Hausdorff, and $Y \subset X$ non-empty and closed, with $\overline{X \setminus Y} = X$. Take $E = A$ and $F = \{f \in A \mid f(Y) = \{0\}\}$. Then

$$F^\perp = \{g \in A \mid \langle f, g \rangle = f^*g = 0, \forall f \in F\}$$

Let $g \in F^\perp$ and assume $|g(x)| > 0$, so there is an open neighbourhood U_x of x where g is non-zero. For $f \in F$ we have $\overline{f(x)}g(x) = 0$, hence $f(U_x) = \{0\}$ for all $f \in F$.

As $X \setminus Y$ is dense in X , there exists a

$$y \in U_x \cap (X \setminus Y).$$

Thus, as X is normal, by Urysohn's lemma there is a continuous function $h : X \rightarrow \mathbb{C}$ with $h(Y) = \{0\}$ and $h(y) = 1$, meaning $h \in F$, but $h(U_x) \neq \{0\}$, a contradiction. So $g = 0$, meaning $F^\perp = \{0\}$.

1.8. Operators on Hilbert C*-Modules

1.8.1. Adjointable Maps

Let E, F be Hilbert A -modules. A function $f : E \rightarrow F$ is *A-linear* if it is linear with $t(xa) = t(x)a$ for all $x \in E$ and $a \in A$.

We define the set of *adjointable* maps $\mathcal{L}(E, F)$ to be a set of maps $t : E \rightarrow F$ with adjoints $t^* : F \rightarrow E$ such that

$$\langle tx, y \rangle = \langle x, t^*y \rangle.$$

We have that each t is A -linear and bounded (by Banach-Steinhaus). The converse does not hold, not every bounded A -linear map has an adjoint.

We write $\mathcal{L}(E) = \mathcal{L}(E, E)$, then $\mathcal{L}(E)$ is a C*-algebra. It is a closed subset of $B(E)$, and thus a Banach algebra. We have

$$\|t^*t\| \geq \sup_{\|x\| \leq 1} \|\langle t^*tx, x \rangle\| = \sup_{\|x\| \leq 1} \|\langle tx, tx \rangle\| = \|t\|^2$$

thus the C*-identity holds.

Proposition 1.5 Let $t \in \mathcal{L}(E, F)$ and $x \in E$, then

$$|tx|^2 \leq \|t\|^2 |x|^2$$

and $|tx| \leq \|t\| |x|$.

Proof. Let ρ be a state on A , then we can define a semi-inner-product on E by

$$\rho(\langle \cdot, \cdot \rangle) : E \times E \rightarrow \mathbb{C}$$

Then for $a \geq 0$, $\rho(a) \leq \|a\|$, so we have

$$\begin{aligned} \rho(\langle t^*tx, x \rangle) &\leq \rho(\langle t^*tx, t^*tx \rangle)^{\frac{1}{2}} \rho(\langle x, x \rangle)^{\frac{1}{2}} \\ &= \rho(\langle (t^*t)^2x, x \rangle)^{\frac{1}{2}} \rho(|x|^2)^{\frac{1}{2}} \\ &\leq \rho(\langle (t^*t)^2x, (t^*t)^2x \rangle)^{\frac{1}{4}} \rho(|x|^2)^{\frac{1}{2} + \frac{1}{4}} \\ &\vdots \\ &\leq \rho(\langle (t^*t)^{2n}x, x \rangle)^{2^{-n}} \rho(|x|^2)^{1-2^{-n}} \\ &\leq \left\| \langle (t^*t)^{2n}x, x \rangle \right\|^{2^{-n}} \rho(|x|^2)^{1-2^{-n}} \\ &\leq \left(\left\| (t^*t)^{2n} \right\| \|x^2\| \right)^{2^{-n}} \rho(|x|^2)^{1-2^{-n}} \\ &\leq \|t^*t\| (\|x\|^2)^{2^{-n}} \rho(|x|^2)^{1-2^{-n}} \end{aligned}$$

taking $n \rightarrow \infty$ we get $\rho(\langle tx, tx \rangle) \leq \|t\|^2 \rho(|x|^2)$. Thus, as ρ was arbitrary $|tx|^2 \leq \|t\|^2 |x|^2$. $\square$

1.8.2. 'Compact' Operators

Now we look at an analogue to compact operators. First we define finite operators. Let E, F be Hilbert A -modules. We define for $x \in E$ and $y \in F$ an element $\theta_{x,y} \in \mathcal{L}(F, E)$ by

$$\theta_{x,y}(z) = x\langle y, z \rangle (z \in F)$$

where $\theta_{x,y}^* = \theta_{y,x}$. Then for $x \in E$, G a Hilbert A -module the following relations hold:

- (i) $\theta_{x,y}\theta_{u,v} = \theta_{x\langle y,u \rangle, v} = \theta_{x,v\langle u,y \rangle}$ ($u \in F, v \in G$)
- (ii) $t\theta_{x,y} = \theta_{tx,y}$ ($t \in \mathcal{L}(E, G)$)
- (iii) $\theta_{x,y}s = \theta_{x,s^*y}$ ($s \in \mathcal{L}(G, F)$)

Then we define $\mathcal{K}(F, E)$ to be the closed linear span of $\{\theta_{x,y} \mid x \in E, y \in F\}$. Again we write $\mathcal{K}(E) = \mathcal{K}(E, E)$, then this is a (closed, two-sided) ideal of $\mathcal{L}(E)$.

It is worth noting that these operators are not necessarily compact as operators between Banach spaces. For A unital, $\theta_{1,1}$ is the identity, which is not compact in general. For A unital we have $\mathcal{K}(A) = \mathcal{L}(A)$.

When $E = A$, we have $\mathcal{K}(E) \cong A$, as

$$\theta_{a,b}(c) = a\langle b, c \rangle = ab^*c$$

[insert proof here]

1.8.3. Tensors

Let H be a Hilbert space. We have $\mathcal{K}(H \otimes A) \cong \mathcal{K}(H) \otimes A$ (for the C*-algebraic tensor product). For $\xi, \eta \in H$ we define the rank-one operator $\xi \cdot \eta$ by

$$\xi \cdot \eta(\zeta) = \xi\langle \eta, \zeta \rangle$$

then $\mathcal{K}(H \otimes A)$ is generated by the elements $(\xi \cdot \eta) \otimes c$.

1.8.4. Matrices

Let E, F be Hilbert A -modules, then $\mathcal{K}(E^m, F^n)$ can be identified with $m \times n$ matrices over $\mathcal{K}(E, F)$. Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in E^m, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in F^n.$$

then we write $\theta_{x,y}$ to be the matrix with θ_{x_i, y_j} entries.

We also identify $\mathcal{L}(E^m, F^n)$ with $m \times n$ matrices over $\mathcal{L}(E, F)$. Associate the element $t \in \mathcal{L}(E^m, F^n)$ with the matrix $(q_j t p_i^*)$ for $p_i^* \in \mathcal{L}(E, E^m)$ the inclusion mapping of E into the i th E^m , and $q_j \in \mathcal{L}(F^n, F)$ the projection of the j th coordinate of F^n onto F

1.9. Topologies

For Hilbert C*-modules we use the *strict* topology. On the space of operators $\mathcal{L}(E, F)$ we define this topology by the semi-norms

$$t \mapsto \|tx\|, \quad t \mapsto \|t^*y\|$$

for all $x \in E$, $y \in F$. Where the topology is given by the smallest topology such that addition, scalar multiplication, and every semi-norm, are continuous. For $t \neq 0$ there is an $x \in E$ such that $\|tx\| > 0$. Thus the topology is Hausdorff.

The following propositions are made use of later in the book

Proposition 1.6 Let E, F be Hilbert A -modules, then the unit ball of $\mathcal{K}(E, F)$ is strictly dense in the unit ball of $\mathcal{L}(E, F)$.

Proposition 1.7 Let E be a Hilbert A -module, and B a strictly dense C*-subalgebra of $\mathcal{L}(E)$. Then the unit ball of B is strictly dense in the unit ball of $\mathcal{L}(E)$.

1.10. A Riesz Representation

This is in the textbook “as a curiosity and an excuse for an exercise”.

Take E a Hilbert A -module, and fix $x \in E$. Define $t_x : E \rightarrow A$ by

$$t_x y \langle x, y \rangle.$$

Thus t_x is a bounded A -linear map. We have

$$\langle t_x y, a \rangle_A = \langle x, y \rangle_E^* a = \langle y, xa \rangle_E.$$

Thus $t_x^*(a) = xa$ is the adjoint of t , meaning $t_x \in \mathcal{L}(E, A)$. We have

$$\theta_{a,z}(y) = a \langle z, y \rangle = \langle za^*, y \rangle = t_{za^*}(y)$$

so $t_x \in \mathcal{K}(E, A)$ when $x = za^*$, but EA is dense in E , and t_x is isometric. Thus we have $t_x \in \mathcal{K}(E, A)$.

Then we think of $\mathcal{K}(E, A)$ as the natural analogue of the set of bounded linear functionals on a Hilbert space. So we get a Riesz representation.

Every $f \in \mathcal{K}(E, A)$ is given by the form t_x for some $x \in E$. The same is not true in general for $\mathcal{L}(E, A)$ but A unital, every element of $\mathcal{L}(E, A) = \mathcal{K}(E, A)$, is given by an inner product. For $t \in \mathcal{L}(E, A)$ we have $t = t_x$ for $x = t^*(1)$.