Arithmetic Dynamics - Second Take

§1.4) The Julia & Fatou Set

Definition (EC): Let (S_1, ρ_1), (S_2, ρ_2) be Metric Spaces and let Φ be a collection of Maps from $S_1 \to S_2$. Then Φ is said to be equicontinuous at a point $\alpha \in S_1$ if for every $\varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$$\rho_1(\alpha, \beta) < \delta \Rightarrow \rho_2(\phi(\alpha), \phi(\beta)) < \varepsilon \quad \forall \phi \in \Phi$$

If Φ is equicontinuous at every point in a subset, then it is equicontinuous on that subset

Definition: For $\phi: S \to S$, ϕ is EC at α if $\{\phi^n : n \geq 1\}$ is equicontinuous at α

Remark: Let $F: X \to X$

Continuity (Pointwise)		$\delta = \delta(\varepsilon, F, \alpha)$	} For a Single Function
Uniform Continuity		$\delta = \delta(\varepsilon, F)$	
Pointwise	EC	$\delta = \delta(\varepsilon, \alpha)$	} For a collection
Uniform	EC	$\delta = \delta(\varepsilon)$	

Examples: 1) $\Phi = \{id\}$ ← Trivial

2) $\Phi = \{F(z)\}$ ← Boring, where $F(z)$ is a Continuous Function

3) $\Phi = \{$Lipschitz Functions with the Same Constant$\}$ $= \{\sqrt{x^2+5}, \sqrt{x^2+4}, x\}$

4) $\Phi = \{\text{arctan}(nx) : n \in \mathbb{N}\}$ Not equicontinuous at 0

Definition: Let ϕ be a Map from a Metric Space to itself.

The Fatou Set of ϕ, denoted $F(\phi)$, is the Maximal open set on which ϕ is EC

Set where nearby points behave the same

The Julia Set of ϕ, denoted $J(\phi)$ is the complement of $F(\phi)$.

Remark: As ϕ is iterated, points in $F(\phi)$ stay together and points in $J(\phi)$ spread apart

Example: Let $\phi(z) \in \mathbb{C}(z)$ be a Rational map.
If α is an attracting periodic point of ϕ, Points near α stay near α under ϕ, so one in $F(\phi)$
If α is Repelling, $\alpha \in J(\phi)$.

Kind of an Answer to Ruths Question $|\lambda_\alpha(\phi)| = 1$?

If $|\lambda_\alpha(\phi)| = 1$ then it's Complicated.

Example: Take $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$
$\phi(z) = z^d$. Then $\phi^n(z) = z^{d^n}$ so

$$\lim_{n\to\infty} \phi^n(\alpha) = \begin{cases} 0 & \text{if } |\alpha| < 1 \\ \infty & \text{if } |\alpha| > 1 \end{cases}$$

So if $|\alpha| \neq 1$, then $\alpha \in F(\phi)$

If $|\alpha| = 1$ then $|\phi^n(\alpha)| = 1$ $\forall\, n \geq 1$ but $\exists\, \beta$ close to α s.t. $\phi^n\beta \to 0$ and $\exists\, \gamma$ s.t. $\phi^n\gamma \to \infty$
$\therefore J(\phi) = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$

$$\mathrm{Per}_n(\phi) = \left\{ e^{2\pi i k/(d^n - 1)} : k \in \mathbb{Z} \right\}$$

Say $\alpha \in \mathrm{Per}_n(\phi)$, $|\lambda_\alpha(\phi)| = |(\phi^n)'(\alpha)| = |d^n \alpha^{d^n - 1}| = d^n > 1$
So periodic points are repelling!

$\mathrm{Per}(\phi) = \bigcup_{n=1}^{\infty} \mathrm{Per}_n(\phi)$ is dense in $J(\phi)$

$J(\phi) = \overline{\text{repelling periodic points of } \phi}$

Proposition 1.24: Let $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be Rational map, deg ≥ 2, Let F & J be the Fatou & Julia sets. Then

a) $\phi^{-1}(F) = F = \phi(F)$

b) $\phi^{-1}(J) = J = \phi(J)$

c) $\phi^{-1}(\partial J) = \partial J = \phi(\partial J)$

Prop 1.25: $\forall\, n \in \mathbb{Z} \geq 1$

$F(\phi^n) = F(\phi)$ & $J(\phi^n) = J(\phi)$

Definition: Let (S_1, ℓ_1) & (S_2, ℓ_2) be metric spaces. Φ a collection of maps from S_1 to S_2.

Then Φ is said to be normal on S_1 if every infinite sequence of functions from Φ contains a subsequence that converges locally uniformly on S_1.

**Example: every point of X has a neighborhood on which F_n converges uniformly to F

Theorem: (IZ, Arzela Ascoli, Complex Analysis) Let U be an open subset of $\mathbb{P}^1(\mathbb{C})$ & Φ a collection of continuous maps. Then Φ is eq on U iff Φ is a normal family on U.

Theorem 1.28 (Montel's): Let U be an open subset of $\mathbb{P}^1(\mathbb{C})$ and let Φ be a collection of analytic maps $U \to \mathbb{P}^1(\mathbb{C})$.

If $\bigcup_{\phi \in \Phi} \phi(U)$ omits 3 or more points of $\mathbb{P}^1(\mathbb{C})$, then Φ is a normal family on U.

Theorem 1.29: Let $\phi(z) \in \mathbb{C}(z)$ a Rational function of deg $d \geq 2$.

a) $J(\phi)$ is non-empty
b) Let $\alpha \in J(\phi)$. Then $O_\phi^-(\alpha) = \{\phi^{-n}(\alpha) : n \geq 1\}$ is dense in $J(\phi)$
c) Let U be the union of the connected components of the Fatou set s.t. $\phi^{-1}(U) = U = \phi(U)$. Then $J(\phi) = \partial U$
d) $J(\phi)$ is perfect

Perfect: contains no isolated points. And closed.

Example: Let $\phi(z) = z^d$. We saw that $|\alpha| > 1$ is one Fatou component & $|\alpha| < 1$ is the other.

Let U be the union clearly $\phi^{-1}(U) = U = \phi(U)$
Thm 1.29 c) $\Rightarrow J(\phi) = \partial U = \{|\alpha| = 1\}$

Example: $\Phi = \{F_m(z) = \frac{m}{m+1} \cdot \frac{1}{z} : m \in \mathbb{N}\}$ on $\{z : 0 < |z| < 1\}$

Semi clearing because bounded

$\Phi = \{\frac{z}{1 - \alpha z} \text{ (Taylor)} : |\alpha| \geq 1\}$ on $\{z : |z| < 1\}$

$\alpha_n \to \infty \Rightarrow F_m \to 0$

or $\{\alpha_n\}$ converges $\Rightarrow$ it has a convergent subsequence $\alpha_n \to \alpha$ then convergence is uniform on a compact subset of $\{z : |z| < 1\}$

$$\left| \frac{z}{z-\alpha_n} - \frac{z}{z-\alpha} \right| = |\alpha_n - \alpha| \frac{|z|}{|z-\alpha_n||z-\alpha|}$$

$$\leq |\alpha_n - \alpha| \frac{|z|}{(1-|z|^2)}$$

So these two families are Normal

Theorem: $\phi(z) \in \mathbb{C}(z)$, $\deg \phi \geq 2$. TFAE

1.20 Conditions For empty Fatou Set

a) $J(\phi) = \mathbb{P}^1(\mathbb{C})$

b) $J(\phi) \setminus \partial J(\phi) \neq \emptyset$

c) $\exists\, \alpha \in \mathbb{P}^1(\mathbb{C})$ s.t. $\mathcal{O}_\phi(\alpha)$ is dense in $\mathbb{P}^1(\mathbb{C})$

d) Every critical point of ϕ is strictly pre-periodic

- α pre-periodic if some iterate $\phi^n(\alpha)$ is periodic i.e. it eventually becomes periodic
- Critical point = Ramification point = α s.t. $\phi'(\alpha) = 0$

Example: $\phi(z) = 1 - \frac{2}{z^2} \Rightarrow \phi'(z) = \frac{1}{z^3}$

This is $\phi'(z) = 0$ when $z = \infty$ and undefined when $z = 0$. So these are the critical points

$$0 \xrightarrow{\phi} \infty \xrightarrow{\phi} 1 \xrightarrow{\phi} -1 \xrightarrow{\phi} -1 \xrightarrow{\phi} \dots$$

So strictly periodic ($z = 0 \in \omega$)

$\Rightarrow$ by Theorem 1.30, $J(z) = \mathbb{P}^1(\mathbb{C})$

Remark: For $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ rational, then it's not possible for $F(\phi) = \mathbb{P}^1(\mathbb{C})$

BUT over different fields we can! E.G.

take $\mathbb{C}_p$ Algebraically closed, non-archimedean absolute value, then it's possible! CHAPTER 2.

§1.5: Properties of Periodic Points

Theorem 1.35: Let $\phi(z) \in \mathbb{C}(z)$, degree $d \geq 2$

Properties/importance of Periodic Points

a) The map ϕ has at most $2d-2$ non-repelling periodic cycles in $\mathbb{P}^1(\mathbb{C})$.

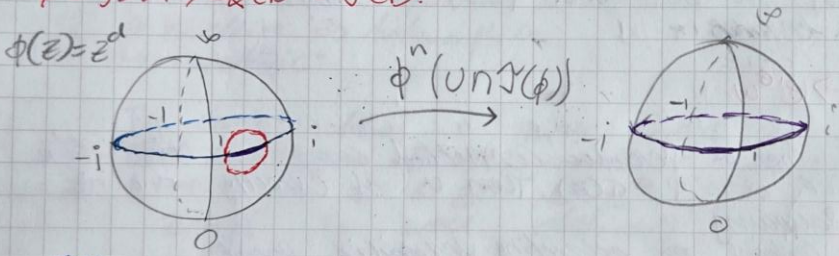
If ϕ is a polynomial map, then it has at most $d-1$ non-repelling periodic cycles in $\mathbb{C}$.

b) $J(\phi) = \overline{\text{repelling periodic points of } \phi}$

c) Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open set s.t. $J(\phi) \cap U \neq \emptyset$. Then there is an integer $n \geq 1$ s.t. $\phi^n(U \cap J(\phi)) = J(\phi)$

Semi-intuitive w/ ϕ being exotic

A rational map is a polynomial map if it has a totally ramified fixed point. That is $\exists\ \alpha \in \mathbb{P}^1(\mathbb{C})$ s.t. $\phi(\alpha) = \alpha$, $e_\alpha(\phi) = \deg(\phi)$.



$J(\phi)$ = Unit Circle

U = an open set $J(\phi) \cap U$

$\phi^n(U \cap J(\phi)) = J(\phi)$

Remark: $F(\phi) \subset \mathbb{P}^1(\mathbb{C})$, open $\therefore$ it consists of $0, 1, 2$, or ∞ connected components (Fatou).

If U is a connected component of $F(\phi)$, then $\phi(U)$ is also a connected component.

U is a periodic domain if $\exists n$ s.t. $\phi^n(U) = U$, preperiodic if not, and non-wandering otherwise

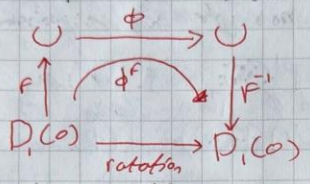
Theorem: [1.36, Sullivan] Non-wandering Domain Theorem: A rational map $\phi \in \mathbb{C}(z)$ has no wandering domains.

1.6) Dynamical Systems Associated to Endomorphisms of Algebraic

Def: Let U be a connected component of $F(\phi)$, assume $U = \phi(U)$ (forward invariant)

Then

- U is parabolic if ∂U contains a rationally neutral periodic point, p, s.t. $\lim_{n\to\infty} \phi^n(\alpha) = p \ \forall \alpha \in U$
- U is a Siegel Disk if there is an analytic isomorphism F from the unit disk to U s.t. ϕ^F is a rotation of the unit disk



$\phi^F = F^{-1} \circ \phi \circ F$

- U is a Herman Ring if there is an analytic isomorphism F from some annulus to U s.t. ϕ^F is a rotation of the annulus

Rotation: $w \mapsto e^{i\theta} w$

Theorem 1.37: Let U be a periodic connected component of the Fatou Set of $\phi \in \mathbb{C}(z)$. Then U is exactly one of the following
- U contains an attracting periodic point
- U is parabolic
- U is a Siegel Disk
- U is a Herman Ring

Ex Multipedia pics

- Attracting Periodic

$$z \xmapsto{\Phi} z - \frac{z^3-1}{3z^2}, \quad Fix = \{z : z^3 = 1\}$$

- Take $z=1 \Rightarrow \lambda_{z_1}(\phi) = 0$ ∴ attracting & periodic

- Parabolic:
  Black line is ∂U,