Reading Group — Arithmetic Dynamics: An introduction to classical Dynamics.

§1.1 Rational Maps & The Projective Line

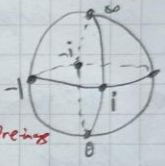
- $\phi(z) \in \mathbb{C}(z)$ is a rational function if it is a Quotient of polynomials

$$\phi(z) = \frac{F(z)}{G(z)} = \frac{a_0 + a_1 z + \dots + a_d z^d}{b_0 + b_1 z + \dots + b_d z^d}$$

with no common factors

- $\deg(\phi(z)) = \max\{\deg(F), \deg(G)\}$
- Following the book, $\deg(\phi) \geq 2$
- If we have a rational function, $\phi: \mathbb{C} \to \mathbb{C}$, then it induces a rational map on the Complex projective line $\mathbb{P}^1(\mathbb{C})$

$$\phi: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$



Riemann sphere

* ↳ ϕ is a ramified d-1 covering z^2, $\deg(z)$ All points, 2-preimages

- $\mathbb{P}^1(\mathbb{C})$ also called Riemann Sphere Topologically, and More Analytically (metrics), $\mathbb{P}^1(\mathbb{C})$ is isomorphic to S^2
- $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ ↗ is this related to Ruth's stuff?

* Ramified: Also called branched. Which means it is a covering map everywhere except for a nowhere dense set.

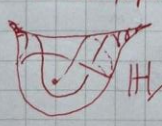
- Ex: let $\phi(z) = z^2$, , $z-1$ branched covering, heuristically z has 2 preimages under $\phi(z)$.

Fundamental theorem of Algebra

Comprised of Singularities.

- $z \mapsto \frac{az+b}{cz+d}$, Möbius, $ad-bc \neq 0$, this gives an automorphism of $\mathbb{P}^1$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $GL(2,\mathbb{R})$ In fact these are the only!
- Cool Fact: If we take instead $z \mapsto \frac{az+b}{cz+d}$ in the same way but for $PSL(2,\mathbb{R})$, then we get the isometry group of or $\mathbb{H}$!!
- Important in Dynamics. This where important/interesting stuff is

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, Hyperbolic Toral Automorphisms



Geodesic Excursions

- Composition of Mobius transformations $\leftrightarrow$ Matrix multiplication.
- Inverse Map is inverse Matrix
- $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = PGL_2(\mathbb{C}) = \underbrace{GL_2(\mathbb{C})/\mathbb{C}^*}_{?}$

- $\exists!\ F(z) \in PGL_2(\mathbb{C})$ s.t. given $(\alpha,\alpha',\alpha'')$, (β,β',β'')

$$F(\alpha)=\beta,\ F(\alpha')=\beta',\ F(\alpha'')=\beta''$$

PF: Find $f \in$ & Find g S.t.

$f(\alpha)=0$ $\quad$ $g(\beta)=0$

$f(\alpha')=1$ $\quad$ $g(\beta')=1$

$f(\alpha'')=\infty$ $\quad$ $g(\beta'')=\infty$

Then just compose with inverse

$F(z_1)=0,\ F(z_2)=1,\ F(z_3)=\infty$

$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

- Notation, $\phi:\mathbb{P}^1\to\mathbb{P}^1$, Rational Map, $F\in PGL_2(\mathbb{C})$

the linear Conjugate of ϕ by F is (Good for iteration)

$$\phi^F = F^{-1}\circ\phi\circ F \Rightarrow (\phi^F)^n = F^{-1}\circ\phi^n\circ F = (\phi^n)^F$$

- we say ϕ & ψ are Linearly Conjugate if $\psi=\phi^F$ for some $F\in PGL_2(\mathbb{C})$ Similar in DS Notion

- Example from Introduction to dynamical Systems!

Ex 1

$M\in\mathbb{Z}$, $|M|>1$, $E_M: S^1\to S^1$

$E_M x = Mx \bmod 1$

Expanding circle Endomorphisms.

$\Sigma = \{0,\dots,M-1\}^{\mathbb{N}}$

$\sigma:\Sigma\to\Sigma$

$\sigma((x_1,x_2,\dots,x_n,\dots)) = (x_2,x_3,\dots)$

$x = \sum_{i=1}^{\infty} \frac{x_i}{m^i}$

Base expansion and Shifts.

sequence

(X,F) $\pi\circ F = g\circ\pi$

(Y,g)

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \exists\pi\downarrow & & \downarrow\pi \\ Y & \xrightarrow{g} & Y \end{array}$$

$\phi:\Sigma\to[0,1]$

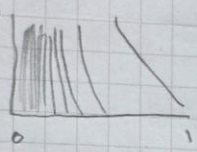
$\phi((x_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty}\frac{x_i}{m_i}$ } Semi Conjugacy Not invertible

Not Conjugacy because $0.0\#\#1\dots = 0.1$

$\phi\circ\sigma = E_M\circ\phi$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi} & [0,1] \\ \sigma\downarrow & & \downarrow E_M \\ \Sigma & \xrightarrow{\phi} & [0,1] \end{array}$$

Ex 2: $\varphi : [0,1] \to [0,1]$, $\varphi(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \in (0,1] \\ 0 & \text{otherwise} \end{cases}$



GAUSS MAP

$[x_1, x_2, x_3 \dots] \xrightarrow{\varphi} [x_2, x_3 \dots]$

$[0,1]_{\text{Number}} \xrightarrow{\varphi} [0,1]_{\text{Number}}$

Continued Fraction

Continued Fraction

$(b_i)_{i=1}^{\omega}, \omega \in \mathbb{N} \cup \{\infty\}, b_i \in \mathbb{N} \xrightarrow{\sigma} (b_i)_{i=1}^{\omega}$

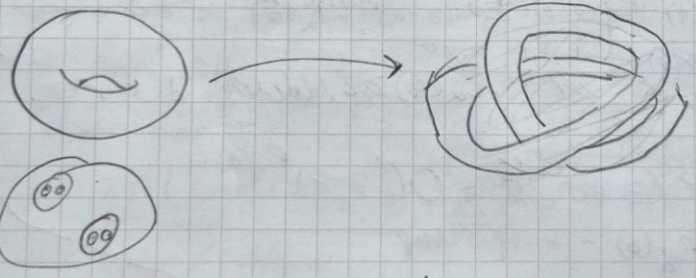
σ ← Shift map

Ex 3: Solenoid:

Solid Torus $T = S^1 \times D^2$, $\lambda \in (0, \frac{1}{2})$

$F : T \to T$

$F(\phi, x, y) = (2\phi, \lambda x + \frac{1}{2}\cos 2\pi\phi, \lambda y + \frac{1}{2}\sin 2\pi\phi)$



But inside the torus

$\Phi = \{(\phi_i)_{i=0}^{\infty} : \phi_i \in S^1, \phi_i = 2\phi_{i+1} \bmod \forall i\}$

$\alpha : \Phi \to \Phi$

$\alpha(\phi_0, \phi_1, \dots) \mapsto (2\phi_0, \phi_0, \phi_1, \dots)$ $\bigcap_{n=0}^{\infty} F^n(T)$

(Φ, α) Conjugate to (S^1, F)

• $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ ... $\iff$ Matrix multiplication

1.2) Critical Points & Riemann Hurwitz Formula

• $\phi'(z) = \dfrac{G(z)F'(z) - F(z)G'(z)}{(G(z))^2}$

Defined $\alpha \neq \infty$, $G(\alpha) \neq 0$

• Taylor Theorem / Approximation

$$\phi(z) = \phi(\alpha) + \phi'(\alpha)(z-\alpha) + O((z-\alpha)^2)$$

• $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is ramified at α if $\phi'(\alpha) = 0$. (or: ramified)
↳ α is called a Critical point
↳ $z_1 = \phi(z_0)$ is a critical value

If $F : \mathbb{C} \to \mathbb{C}$ is ... $F' = 0$, then it is not locally injective at that point.
(Inverse Function Theorem)

• Assume $\alpha \neq \infty$, $\phi(\alpha) \neq \infty$, the ramification index of ϕ at α is

$$e_\alpha(\phi) = \mathrm{ord}_\alpha(\phi(z) - \phi(\alpha)) = \mathrm{ord}_\alpha(\phi'(\alpha)(z-\alpha) + O((z-\alpha)^2))$$

• "$e_\alpha(\phi)$ is the number s.t. locally about the point α, ϕ is $z \mapsto z^{e_\alpha(\phi)}$ (branch point)" $\left(\phi(z) = \sum_{i=0}^{\infty} a_i (z-\alpha)^i$, $\mathrm{ord}_\alpha(\phi)$ is the smallest i s.t. $a_i \neq 0\right)$

• ϕ is ramified at α iff $e_\alpha(\phi) \geq 2$. Locally $\exists\ c \neq 0$ s.t.

$$\phi(z) = \phi(\alpha) + c(z-\alpha)^{e_\alpha(\phi)} + O((z-\alpha)^{e_\alpha(\phi)+1})$$

• Locally $\phi(z)$ is $e_\alpha(\phi)$-to-1 map

• $e_\phi(\alpha) \leq \deg(\phi)$

• $e_\phi(\alpha) = \deg(\phi)$, then we say ϕ is totally ramified at α.

• Theorem: $2d-2 = \sum_{\alpha \in \mathbb{P}^1} (e_\alpha(\phi) - 1)$

where d is the degree of the covering map.

Corr 1.2: A rational map of degree d has exactly $2d-2$ critical points counted with multiplicity.

We can get a weaker version of Riemann-Hurwitz:

Cor 1.3) Let $\phi: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 1$.

a) Let $\alpha \in \mathbb{P}^1(\mathbb{C})$. Then

$$\sum_{\beta \in \phi^{-1}(\alpha)} e_\beta(\phi) = d$$

b) Weak Riemann-Hurwitz Formula

$$2d-2 = \sum_{\alpha \in \mathbb{P}^1(\mathbb{C})} (d - \#\phi^{-1}(\alpha))$$

"weaker because it is just counting pre-images"

The other is defined using ~~Examples~~ ramification index, captures the local behaviour everywhere and has more of the covering map"

"see this number ~~$2d-2$~~ as the total discrepancy across ramification points or the degree and how branched at that point the map is"

- Local Global Relationship

- And Again RH. C_1, C_2 be algebraic curves (Riemann Surfaces) of genus g_1, g_2. Let $\phi: C_1 \to C_2$, finite map of degree $d \geq 1$ Then

$$2g_1 - 2 = d(2g_2 - 2) + \sum_{P \in C_1} (e_P(\phi) - 1)$$

Global / Topological Quantity

Local / Dynamical Quantity

Sullivan: $\limsup_{z \to \infty} \frac{d(z, \phi(z))}{\log(z)} = \frac{1}{d}$ along geodesic flow

Theorem 1.6: $\phi: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, rational, deg $d \geq 2$. Let $E \subset \mathbb{P}^1(\mathbb{C})$ be a finite set s.t. $\phi^{-1}(E) = E$. Then $\#E \leq 2$

a) If $\#E = 1$, then $\exists\, F \in PGL_2(\mathbb{C})$ s.t. $E = \{F(\infty)\}$ & $\phi^F(z) \in \mathbb{C}[z]$. ← Polynomials over $\mathbb{C}$

b) $\#E = 2$, $\exists\, F \in PGL_2(\mathbb{C})$ s.t.

$E = \{F(0), F(\infty)\}$, $\phi^F(z) = z^d$ or z^{-d}.

Dynamical Systems - Invariant Sets - N

- Composition of Möbius transformations $\Leftrightarrow$ Matrix multiplication
- Inverse map is inverse matrix

- A rational map is a polynomial map if it has a totally ramified fixed point
  $\exists \alpha \in \mathbb{P}^1$ s.t. $\phi(\alpha)=\alpha$, $e_\alpha(\phi)=\deg(\phi)$

- If E satisfies $\phi^{-1}(E)=E=\phi(E)$, then E is called completely invariant for ϕ.
  ↳ Really though we only need $\phi^{-1}(E)=E$, and the surjectivity of ϕ

Thm 1.7:
- let $\phi:\mathbb{P}^1(\mathbb{C})\to\mathbb{P}^1(\mathbb{C})$ be rational map, $\deg \geq 2$, and suppose ϕ^n is a polynomial map for some $n\geq 1$. Then ϕ^2 is polynomial map

- If ϕ is not polynomial, then ϕ is linear conjugate to $\frac{1}{z^d}$. (By Möbius Transformation from $PGL_2(\mathbb{C})$)

§ 1.3) Periodic Points and Multipliers

- α be a fixed point of ϕ, then the multiplier of ϕ at α is
  $$\lambda_\alpha(\phi)=\phi'(\alpha)$$
  ($\alpha=\infty$, then modification required) $(\lambda_\infty(\phi)=(\phi^f)'(f^{-1}(\infty)))$

"Does this not depend on the co-ordinates?"

Prop 1.9: let $\phi\in\mathbb{C}(z)$, rational map $\phi(\alpha)=\alpha$. let $f\in PGL_2\mathbb{C}$ be a change of co-ordinates. let $\beta=f^{-1}(\alpha)$. Now β is a fixed point of $\phi^f=f^{-1}\circ\phi\circ f$. Then
$$\phi'(\alpha)=(\phi^f)'(\beta)$$

Pf: Chain rule

∴ The multiplier is well defined

- What about points fixed by some iterate of ϕ?

- Def: - α is called periodic if $\phi^n(\alpha)=\alpha$ for some n.
  - The smallest n is called the exact period of α "First return time"
  - $\mathrm{Per}_n(\phi)=\{\alpha\in\mathbb{P}^1(\mathbb{C}):\phi^n(\alpha)=\alpha\}$

$\phi=z^2$, $\mathrm{Per}_3(\phi)=\{0,\text{roots of } \frac{1}{z^3-1}\}$ (roots of unity)
$\mathrm{Per}_1(\phi)=\{0,1,\infty\}$
$\phi=z^2+1$, $\mathrm{Per}_1(\phi)=\left\{\frac{1\pm\sqrt{-3}}{2}\right\}$ Fixed Points

For $\phi=z^d$, $\mathrm{Per}_3(\phi)=\mathrm{Per}_3^{**}(\phi)$

$$\mathrm{Per}_n^{**}(\phi)=\{\alpha\in\mathrm{Per}_n(\phi):\alpha \text{ has exact Period } n\}$$

$$\Rightarrow \mathrm{Per}_n(\phi)=\bigsqcup_{m\mid n}\mathrm{Per}_m^{**}(\phi)$$

So if n is prime, then $\mathrm{Per}_n=\mathrm{Per}_n^{**}(\phi)$

- Let $\alpha \in Per_n^{**}(\phi)$. Then $\lambda_\alpha(\phi) := (\phi^n)'(\alpha)$

which by chain rule

$$(\phi^n)'(\alpha) = \phi'(\alpha)\cdot\phi'(\phi\alpha)\cdot\phi'(\phi^2\alpha)\cdots\phi'(\phi^{n-1}\alpha)$$

- Remark: $\{\lambda_\alpha(\phi) : \alpha \in Per(\phi)\}$ only depends on the conjugacy class $\{\phi^F : F \in PGL_2(\mathbb{C})\}$ of ϕ

- Let α be a periodic point of $\phi \in \mathbb{C}(z)$, rational. Then

α is called

super attracting	if	$\lambda_\alpha(\phi) = 0$
attracting	if	$\lvert\lambda_\alpha(\phi)\rvert < 1$
neutral	if	$\lvert\lambda_\alpha(\phi)\rvert = 1$
repelling	if	$\lvert\lambda_\alpha(\phi)\rvert > 1$

↗ Also applies to p-adic dynamics

- Thm 1.14: Let k be an algebraically closed field, $\phi(z) \in k(z)$ a rational function of degree $d \geq 2$. Assume

$$\lambda_P(\phi) \neq 1 \quad \forall P \in Fix(\phi)$$

Then $\displaystyle\sum_{P \in Fix(\phi)} \frac{1}{1-\lambda_P(\phi)} = 1$

Ex 1) $\frac{z^3}{z^2-1} : 3z+2z, Fix = \{3, 2, -4, \infty\}$
$\lambda_\infty = 0,\ \lambda_3 = 8,\ \lambda_2 = -5,\ \lambda_{-4} = \frac{1}{3}$

Non Ex 2) $\frac{z^2+2z-1}{z}, \{1, -1, \infty\} = Fix\ z$
$\lambda_\infty = 1$

indication

↳ Dynamics in this setting are characterised by local behaviour of the fixed points.

↳ There's a balance between attracting & repelling regions

We can do something similar for periodic n-points

Ex: $\phi(z) = z^2 + c$

$$\alpha = \frac{1+\sqrt{1-4c}}{2},\quad \beta = \frac{1-\sqrt{1-4c}}{2},\quad \infty$$

Ex 1.13

$$\lambda_\alpha(\phi) = 2\alpha,\quad \lambda_\beta(\phi) = 2\beta,\quad \lambda_\infty(\phi) = 0$$

$$\Rightarrow \frac{1}{1-2\alpha} + \frac{1}{1-2\beta} + \frac{1}{1-0} = 1$$

So we can use this theorem to figure out more difficult pairs multiplier given that we have the others.

composition of Mobius transformations

Sketch of Proof of Thm: 1.14

- Assume k has characteristic 0
- $\therefore$ Lefschetz principle applied (says we can embed a sufficiently large subfield of k into $\mathbb{C}$). So we only need to worry about $k = \mathbb{C}$.
- Make a change of variables (conjugate ϕ by $f \in PGL_2(\mathbb{C})$) s.t. $\phi(\infty) \neq \infty$

- Cauchy Residue theorem: $\forall$ rational function $\psi(z) \in \mathbb{C}(z)$, (more generally meromorphic function).

$$\sum_{P \in \mathbb{P}^1(\mathbb{C})} \operatorname{Res}_P (\psi(z)dz) = 0$$

For $P = \alpha \in \mathbb{C}$ in the affine plane;

$$\operatorname{Res}_{z=\alpha}(\psi(z)dz) = \int_{|z-\alpha|=\varepsilon} \psi(z)dz \text{, for sufficiently small } \varepsilon > 0.$$

If ε large it may encompass another isolated singularity ($|z-\alpha| = \varepsilon$

- $P = \infty$, substitute $z = \frac{1}{w}$, use the same formula to compute Residue w.r.t. w at $\alpha = 0$.

- we apply Cauchy Residue formula to the rational function

$$\psi(z) = \frac{1}{\phi(z) - z}$$

- Poles of $\psi(z) = \operatorname{Fix}(\phi)$

- $\forall \alpha \in \operatorname{Fix}(\phi)$ (Note $\alpha \neq \infty$) we see

$$\begin{aligned}\phi(z) - z &= \left(\phi(\alpha) - \lambda_\alpha(\phi)(z-\alpha) + O(z-\alpha)^2\right\} - z \quad \text{Taylor Expansion} \\ &= (\alpha + \lambda_\alpha(\phi)(z-\alpha) + O(z-\alpha)^2) - z \\ &= (z-\alpha)\left((\lambda_\alpha(\phi) - 1) + O(z-\alpha)\right\}\end{aligned}$$

$\alpha \in \operatorname{Fix}(\phi)$

- $\therefore$ simple pole at $z = \alpha$

- $\operatorname{Res}_{z=\alpha}\left(\frac{dz}{\phi(z)-z}\right) = \frac{1}{\lambda_\alpha(\phi) - 1}$

- $\frac{dz}{\phi(z)-z}$ has no other poles

- So we only need to figure out Res at ∞.
- Substitute $z = \frac{1}{w}$, $d(w^{-1}) = -w^{-2}dw$
- Assumption $\phi(\infty) \neq \infty$ $\Rightarrow$ $w\phi(w^{-1})$ vanishes at $w=0$. $\therefore$

$$\operatorname*{Res}_{z=\infty}\left(\frac{dz}{\phi(z)-z}\right) = \operatorname*{Res}_{w=0}\left(\frac{-dw}{w(w\phi(w^{-1})-1)}\right) = \lim_{w=0} w\left(\frac{-1}{w(w\phi(w^{-1})-1)}\right) = 1$$

- Substituting Both of these gives:

$$0 = \sum_{p \in \mathbb{P}^1(\mathbb{C})} \operatorname*{Res}_{p}\left(\frac{dz}{\phi(z)-z}\right) = \sum_{\alpha \in \mathrm{Fix}(\phi)} \operatorname*{Res}_{z=\alpha}\left(\frac{dz}{\phi(z)-z}\right) + \operatorname*{Res}_{z=\infty}\left(\frac{dz}{\phi(z)-1}\right)$$

$$\Rightarrow 0 = \sum_{\alpha \in \mathrm{Fix}(\phi)} \left(\frac{1}{\lambda_\alpha(\phi)-1}\right) + 1$$