The construction of spectral sequences.

Three ways spectral sequences are usually constructed

double complex $\rightsquigarrow$ filtration $\rightsquigarrow$ exact couple

the most general one is the exact couple method.

Def.: An exact couple $\mathcal{E}$ is a pair (D, E) of modules, together with morphisms i, j, and k s.t.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is exact at each vertex of the triangle.

Def.: The derived couple

$$\begin{tikzcd} i(D) \arrow[rr, "i'"] & & i(D) \arrow[dl, "j'"] \\ & H(E) \arrow[ul, "k'"] & \end{tikzcd}$$

of an exact couple

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is obtained as follows:

$$H(E) \cong \ker(jk)/\mathrm{im}(jk) \quad \text{as}$$

$$(jk)(jk) = j(kj)k) = 0$$

$$i' = i|_{i(D)}$$

$$j'(i(d)) = [j(d)]$$

$$k'([e]) = k(e)$$

we check: ① $j(d)$ is a cycle: $j\underbrace{kj}_{0}(d)=0$

② $\underbrace{i(d)=i(d')}_{\Downarrow} \Rightarrow j(d)-j(d') \in im(jk)$

$d-d' \in ker(i) \Rightarrow d-d' \in im(k) \Rightarrow j(d-d') \in j(im(k))$

①&② $\Rightarrow$ j' is well-defined

③ $\forall e \in ker(jk)$ $\quad k(e) \in i(D)$

$\Rightarrow k(e) \in ker(j) = im(i)$

④ $\forall e \in im(jk)$ $\quad k(e)=0$

$k(jk(e')) = (\underbrace{kj}_{0})k(e') = 0$

③&④ $\Rightarrow$ k' is well-defined.

<u>Claim:</u> $\mathcal{E}'$ is also an exact couple.

<u>Proof:</u> $im(k') \subseteq ker(i')$: $\quad i'(k'([e])) = ik(e) = 0$

$im(j') \subseteq ker(i')$: $\quad j'(i(i(d))) = j(i(d)) = 0$

$im(j') \subseteq ker(k')$: $\quad k'([j(d)]) = k(j(d)) = 0$

$ker(i') \subseteq im(k')$: $\quad i(i(d)) = 0 \Rightarrow i(d) \in ker(i) = im(k)$

$\exists e \in E$ s.t. $k(e) = i(d) \quad jk(e) = ji(d) = 0$

$\Rightarrow k'([e]) = i(d)$

$ker(j') \subseteq im(i')$: $\quad j'(i(d)) = [0] \in H(E) \Rightarrow j(d) = im(jk)$

$j(d) = jk(e) \Rightarrow d - k(e) \in ker(j) = im(i)$

$\exists d'$ s.t. $i(d') = d - k(e)$

now $i(i(d')) = i(d) - 0 \Rightarrow i(d) \in im(i')$.

$\ker(k') \subseteq \mathrm{im}(j')$: $k'([e]) = 0 \Rightarrow k(e) = 0 \Rightarrow \exists\, d$ s.t. $j(d) = e$
$\Rightarrow j'(i(d)) = [e]$ $\square$

We can therefore iterate the process and get the r^{th} derived couple $\mathcal{E}^r$ of $\mathcal{E}$:

$$\mathcal{E}^r = \begin{tikzcd} D^r \arrow[rr, "i"] & & D^r \arrow[dl, "j^{(r)}"] \\ & E^r \arrow[ul, "k"] & \end{tikzcd}$$

and $D^r = i^r(D) \subseteq D$
$E^r = H(E^{r-1})$
i & k are induced from i & k of $\mathcal{E}$
$j^{(r)}(i^r(d)) = [j(d)]$

We will now assume that D and E are bigraded modules,
i has bidegree $(1,-1)$
k has bidegree $(-1,0)$
j has bidegree $(-a,a)$

Set $D'_{p,q} = i(D_{p-1,q+1}) \subseteq D_{p,q}$, $E'_{p,q}$ to be a subquotient of $E_{p,q}$.

$$\begin{tikzcd} D_{p,q} \arrow[r, "j"] & E_{p-a,q+a} \arrow[r, "k"] & D_{p-a-1,q+a} \\ D_{p-1,q+1} \arrow[u, "i"] \arrow[r, "j"] \arrow[urr, dotted, "j'"] & E_{p-a-1,q+a+1} & \end{tikzcd}$$

i' has bidegree $(1,-1)$
k' has bidegree $(-1,0)$
j' has bidegree $(-a-1, a+1)$

$j'(i(d)) = j(d)$

We can reindex so that $\mathcal{E}^r$ is the $(r-a)^{th}$ derived couple of $\mathcal{E}^a$ so we get bideg$(j) = (-r, r)$.

Consider $d^r = j^{(r)}k : E^r_{p,q} \longrightarrow E^r_{p-r+1,q+r}$. We have now constructed a spectral sequence $\{(E^r_{p,q}, d^r)\}$ and a morphism of exact couples now induces a morphism of spectral sequences.

To separate the grading we may consider the following diagram

assume, that $a = 0$:

$$\begin{tikzcd}
D^0_{p-1,q} \arrow[drr, "i"] & E^0_{p,q} \arrow[l, "k"'] & D^0_{p,q} \arrow[l, "j"'] \arrow[drr, "i"] & E_{p+1,q} \arrow[l, "k"'] & D^0_{p+1,q} \arrow[l, "j"'] \\
D^0_{p-1,q-1} \arrow[drr, "i"] & E_{p,q-1} \arrow[l, "k"'] & D_{p,q-1} \arrow[l, "j"'] \arrow[drr, "i"] & E_{p+1,q-1} \arrow[l, "k"'] & D_{p+1,q-1} \arrow[l, "j"'] \\
D_{p-1,q-2} & E_{p,q-2} \arrow[l, "k"'] & D_{p,q-2} \arrow[l, "j"'] & E_{p+1,q-2} \arrow[l, "k"'] & D_{p+1,q-2} \arrow[l, "j"']
\end{tikzcd}$$

we have these LES's spliced together

for $r = 1$

$$\begin{tikzcd}
D^1_{p-1,q} \arrow[drr, "i"] & E^1_{p,q} \arrow[l, "k"'] & D^1_{p,q} \arrow[l, "k"', phantom] \arrow[drr, "i"] & E^1_{p+1,q} \arrow[l, "k"'] & D^1_{p+1,q} \arrow[drr, "i"] & E^1_{p+2,q} \arrow[l, "k"'] & D_{p+2,q} \\
D^1_{p-1,q-1} \arrow[drr, "i"] & E^1_{p,q-1} \arrow[l, "k"'] & D^1_{p,q-1} \arrow[drr, "i"] & E^1_{p+1,q-1} \arrow[l, "k"'] & D^1_{p+1,q-1} \arrow[ulll, "j"] \arrow[drr, "i"] & E^1_{p+2,q-1} \arrow[l, "k"'] & D^1_{p+2,q-1} \arrow[ulll, "j"] \\
D^1_{p-1,q-2} & E^1_{p,q-2} \arrow[l, "k"'] & D^1_{p,q-2} & E^1_{p+1,q-2} \arrow[l, "k"'] & D^1_{p+1,q-2} \arrow[ulll, "j"] & E_{p+2,q-2} \arrow[l, "k"'] & D^1_{p+2,q-1} \arrow[ulll, "j"]
\end{tikzcd}$$

Proposition: Assume, that

$\forall n \; \exists f(n) \geq 0$ s.t. $\forall p,q \; (|p| \geq f(n) \wedge p+q=n) \Rightarrow i : D_{p,q} \to D_{p+1,q-1}$ is an isomorphism

This means, that by exactness $k : E_{p-1,q} \to D_{p,q}$ and $j : D_{p+1,q-1} \to E_{p-a+1,q+a-1}$ are zero. Furthermore in the LES once

$$\begin{tikzcd}
\cdots & D_{p-a,q+a-1} \arrow[d, "i"] & E_{p-a+1,q+a-1} \arrow[l, "k"'] & D_{p,q} \arrow[d, "i"] & E_{p+1,q} \arrow[l, "k"'] & \\
& D_{p-a+1,q+a-2} \arrow[ul, "j"] & & D_{p+1,q-1} \arrow[ul, "j"] & & D_{p+a+1,q-a} \arrow[ul, "j"]
\end{tikzcd}$$

if $p < -N$ then the i's to the left are all isomorphisms, if $p > N$, then all the i's to the right are isomorphisms

this implies, that by exactness

$$\forall n \; \exists r_0 \text{ s.t. } \forall p,q,r \; \left((r \geq r_0 \wedge p+q=n) \Rightarrow \underbrace{j^r_{p-1,q} k^r_{p,q}}_{d^r_{p,q}} = 0 \right).$$

Therefore, $E^\infty_{p,q} = E^r_{p,q}$ for large enough r. Similarly denote $H^\infty_{p+q} = D_{p,q}$ for $p \geq f(n)$ $\quad H^{-\infty}_{p+q} = D_{p,q}$ for $p \leq -f(n)$.

1 If $\forall n \; H_n^{-\infty} = 0$, then $E^\infty_{p,q} \cong F^p_{p+q} / F^{p-1}_{p+q}$ where $\ldots \subseteq F^{p-1}_{p+q} \subseteq F^p_{p+q} \subseteq \ldots$ is a filtration of H^∞_{p+q} by the subgroups $F^p_{p+q} = \operatorname{im}(i^\infty : D_{p,q} \to H^\infty_{p+q})$

② If $\forall n$ $H_n^\infty = 0$, then $E^\infty_{p,q} \cong \tilde{F}^{p+q-1}_p / \tilde{F}^{p+q-1}_{p-1}$ where

$\dots \subseteq \tilde{F}^{p+q-1}_{p-1} \subseteq \tilde{F}^{p+q-1}_p \subseteq \dots$ is a filtration of $H^{-\infty}_{p+q-1}$ by the subgroups $\tilde{F}^{p+q-1}_p = \ker(H^{-\infty}_{p+q-1} \xrightarrow{i^{-\infty}} D^a_{p,q-1})$.

<u>Proof</u>: Consider the exact sequence

$$\begin{tikzcd}
D^r_{p+r-1,q-r+1} \arrow[r, "i"] & D^r_{p+r,q-r} \arrow[r, "j^{(r)}"] & E^r_{p,q} \arrow[r, "k"] & D^r_{p-1,q} \arrow[r, "i"] & D^r_{p,q-1} \\
D_{p-1,q+1} \arrow[u, "i^r"] \arrow[r, "i"] \arrow[ur, phantom, "\circlearrowleft"] & D_{p,q} \arrow[u, "i^r"] & & D_{p-r-1,q+r} \arrow[u, "i^r"] \arrow[r, "i"] \arrow[ur, phantom, "\circlearrowleft"] & D_{p-r,q+r-1} \arrow[u, "i^r"]
\end{tikzcd}$$

Under assumption ① for large enough r, $D_{p-r-1,q+r} = 0$

so $D^r_{p-1,q} = 0$, and

$$D^r_{p+r-1,q-r+1} = \mathrm{im}\,(D^a_{p-1,q+1} \xrightarrow{i^\infty} H^\infty_{p+q}) = F^{p+q}_{p-1}$$
$$D^r_{p+r,q-r} = F^{p+q}_p$$

$$\begin{tikzcd}
D_{p-1,q+1} \arrow[r, "i"] \arrow[drrr, two heads, bend right, "i^\infty"'] & D_{p,q} \arrow[r] \arrow[drrr, two heads, bend right, "i^\infty"] & \cdots \arrow[r] & D_{p+r-1,q-r+1} \arrow[r, "\cong"] \arrow[d, phantom, sloped, "\supseteq"] & D_{p+r,q-r} \arrow[d, phantom, sloped, "\supseteq"] \\
 & & & D^r_{p+r-1,q-1+r} \arrow[r, hook] & D^r_{p+r,q-r}
\end{tikzcd}$$

Under assumption ② for large enough r, $D_{p+r,q-r} = 0$ and

$$\begin{tikzcd}
D_{p-r-1,p+r} \arrow[r, "\cong"] \arrow[drrr, two heads, bend right, "i^{-\infty}"'] & D_{p-r,p+r-1} \arrow[r] \arrow[drrr, two heads, bend right, "i^{-\infty}"] & \cdots \arrow[r] & D_{p-1,q} \arrow[r] \arrow[d, phantom, sloped, "\supseteq"] & D_{p,q-1} \arrow[d, phantom, sloped, "\supseteq"] \\
 & & & D^r_{p-1,q} \arrow[r, two heads] & D^r_{p,q-1}
\end{tikzcd}$$

So we get a SES:

$$0 \to E^r_{p,q} \to D^r_{p-1,q} \xrightarrow{i} D^r_{p,q-1} \to 0$$

Now we apply the Snake lemma:

$$\begin{tikzcd}
 & 0 \arrow[r] & \check{F}^{p+q-1}_{p-1} \arrow[r] \arrow[d] & \tilde{F}^{p+q-1}_{p} \arrow[d] \arrow[dddll, bend left, out=0, in=180] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^{-\infty}_{p+q-1} \arrow[r, equal] \arrow[d] & H^{-\infty}_{p+q-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^r_{p,q} \arrow[r] \arrow[d, equal] & D^r_{p-1,q} \arrow[r] \arrow[d] & D^r_{p,q-1} \arrow[r] & 0 \\
 & E^r_{p,q} \arrow[r] & 0 & &
\end{tikzcd}$$

and we get $E^r_{p,q} \cong \tilde{F}^{p+q-1}_p / \tilde{F}^{p+q-1}_{p-1}$ $\square$

<u>The exact couple of a filtration</u>

Let C be a filtered chain complex and consider the bigraded homology modules

$$D^1_{p,q} \cong H_n(F_p C)$$

$$E^1_{p,q} \cong H_n(F_pC/F_{p-1}C) \quad \text{where } n = p+q.$$

The short exact sequences $0 \to F_{p-1} \to F_p \to F_p/F_{p-1} \to 0$ give long exact sequences on homology and we can splince into the diagram

$$\begin{tikzcd}
\cdots & H_{p+q-1}(F_{p-1}C) \arrow[d, "i_{p-1,q}"] & H_{p+q}(F_pC/F_{p-1}C) \arrow[l, "k_{p,q}"'] & H_{p+q}(F_pC) \arrow[d, "i_{p,q}"] & \cdots \arrow[l] \\
\cdots & H_{p+q-1}(F_pC) \arrow[d, "i_{p,q-1}"] \arrow[ul] & H_{p+q}(F_{p+1}C/F_pC) \arrow[l, "k_{p+1,q-1}"] & H_{p+q}(F_{p+1}C) \arrow[d, "i_{p+1,q-1}"] \arrow[ul, "j_{p+1,q-1}"'] & \cdots \arrow[l] \\
 & H_{p+q-1}(F_{p+1}C) \arrow[ul] & & H_{p+q}(F_{p+2}C) \arrow[ul, "j_{p+2,q-2}"'] &
\end{tikzcd}$$

Which we can roll up as

$$\begin{tikzcd}
\mathcal{E}^1 : \bigoplus_{p,q} H_{p+q}(F_p C) \arrow[rr, "i"] & & \bigoplus_{p,q} H_{p+q}(F_p C) \arrow[dl, "j"] \\
& \bigoplus_{p,q} H_{p+q}(F_p C / F_{p-1} C) \arrow[ul, "k"] &
\end{tikzcd}.$$

Now assuming, that the filtration is bounded, i.e. for small enough p $F_p C = 0$, for large enough p $F_p C = C$, we get that the spectral sequence of this exact couple converges:

$$E^r_{p,q} \Rightarrow H_{p+q}(C).$$

Consider a double complex C, now we have two filtrations of $\mathrm{Tot}(C)$ that we can consider: a vertical and a horizontal one!

Let ${}^I F_n \mathrm{Tot}(C) = \mathrm{Tot}({}^I \tau_{\leq n} C)$ where

$$({}^I\tau_{\leq n} C)_{p,q} = \begin{cases} C_{p,q} & \text{if } p \leq n \\ 0 & \text{if } p > n \end{cases}$$

so $$({}^I F_n \mathrm{Tot}(C))_m = \bigoplus_{\substack{p+q=m \\ p \leq n}} C_{p,q}.$$

The spectral sequence of this is ${}^I E_{\cdot,\cdot}$.

$${}^I E^0_{p,q} \cong ({}^I F_p \mathrm{Tot}(C))_{p+q} / ({}^I F_{p-1} \mathrm{Tot}(C))_{p+q} \cong \left(\bigoplus_{\substack{a+b=p+q \\ a \leq p}} C_{a,b} \right) \Big/ \left(\bigoplus_{\substack{a+b=p+q \\ a \leq p-1}} C_{a,b} \right) \cong C_{p,q}$$

the d^0 differentials are exactly those that are induced by
$$d^v: C_{*,\bullet} \to C_{*-1,\bullet}.$$
Let us consider what are the d^1 differentials:

$$d^1: H^v_q(C_{p,*}) \longrightarrow H^v_q(C_{p-1,*})$$

$$\begin{tikzcd}[column sep=small, row sep=small]
\ddots & C_{p-2,q+1} \arrow[d, "d^v"] & & \\
\ddots & C_{p-2,q+1} & C_{p-1,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \\
& & C_{p-1,q} & C_{p,q} \arrow[l, "d^h"'] \arrow[d] & \ddots \\
& & & C_{p,q-1} & \ddots
\end{tikzcd}$$

now we have "quotiented out $d^v + d^h$ by d^v"
so we are left with d^h!

We can similarly take the horizontal filtration yielding ${}^{II}E_{p,q}$ with ${}^{II}E^0_{p,q} = C_{q,p}$ and ${}^{II}E^1_{p,q} = H^h_q(C_{*,p})$.

<u>Balancing Tor</u>

Let $P_\bullet \to A$ be a projective resolution of A and
$Q_\bullet \to B$ a projective resolution of B.
We get a double complex $C_{\bullet,\bullet}$ via the sign trick:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A \otimes Q_2 \arrow[d] & P_0 \otimes Q_2 \arrow[d, "d"'] & P_1 \otimes Q_2 \arrow[l, "d"'] \arrow[d, "-d"'] & \cdots \arrow[l, "d"'] \arrow[d] \\
A \otimes Q_1 \arrow[d] & P_0 \otimes Q_1 \arrow[d, "d"'] & P_1 \otimes Q_1 \arrow[l, "d"'] \arrow[d, "-d"'] & P_2 \otimes Q_1 \arrow[l, "d"'] \arrow[d, "d"'] & \cdots \arrow[l] \\
A \otimes Q_0 & P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l, "d"'] & P_2 \otimes Q_0 \arrow[l, "d"'] & \cdots \arrow[l] \\
& P_0 \otimes B & P_1 \otimes B \arrow[l, "d"'] & P_2 \otimes B \arrow[l, "d"'] & \cdots \arrow[l]
\end{tikzcd}$$

Since each Q_i and P_i is projective tensoring with them is exact:

$${}^{I}E^2_{p,q} \cong \begin{cases} H^h_p(P\otimes B) = L_p(-\otimes B)(A) & \text{if } q=0 \\ 0 & \text{otherwise} \end{cases}$$

$${}^{II}E^2_{pq} \cong \begin{cases} H^v_p(A\otimes Q) = L_p(A\otimes -)(B) & \text{if } q=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathrm{Tor}_p(A,B) \cong L_p(-\otimes B)(A) \cong L_p(A\otimes -)(B)$$

Base-change for Tor

Theorem: Let $f: R \to S$ be a ring map. Then there is a first quadrant homology spectral sequence

$$E^2_{p,q} \cong \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A,S), B) \Rightarrow \mathrm{Tor}^R_{p+q}(A,B)$$

$\forall A \in \mathrm{Mod}_R$, $B \in {}_S\mathrm{Mod}$.

Proof: Let $P. \to A$ be a projective resolution of A in Mod_R.

Let $Q. \to B$ be a projective resolution of B in ${}_S\mathrm{Mod}$.

We have a restriction of scalars functor $f^*: {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$

So we have a double complex of abelian groups

$$P. \otimes_R Q. := P. \otimes_R f^* Q.$$

We write $H_*(P\otimes_R Q)$ for $H_*(\mathrm{Tot}(P.\otimes_R f^*(Q.)))$.

Since $P\otimes -$ is exact ${}^{I}E^1_{p,q} \cong H^v_q(P_p \otimes_R f^*(Q.))$ furthermore, f^* is exact yielding

$$H_*(P\otimes f^*B) \cong \mathrm{Tor}^R_*(A,B).$$

This means, that ${}^{II}E$ also converges to $\mathrm{Tor}^R_*(A,B)$.

$${}^{II}E^1_{p,q} = H_q(P_\bullet \otimes_R Q_p) = H_q((P_\bullet \otimes_R S) \otimes_S Q_p)$$

$$= H_q(P \otimes_R S) \otimes_S Q_p = \mathrm{Tor}^R_q(A,S) \otimes_S Q_p$$

hence

$${}^{II}E^2_{p,q} \cong \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A,S), B). \qquad \square$$

<u>Constructing the Leray-Serre spectral sequence</u>

Remember we had a fiber sequence $F \to X \xrightarrow{\pi} B$. Assuming, that B is a CW complex, we get a filtration of X via $X_p = \pi^{-1}(sk_p(B))$. Now we get the following diagram of spliced together LESs:

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & H_{n+1}(X_p) \arrow[r] \arrow[d] & H_{n+1}(X_p, X_{p-1}) \arrow[r] & H_n(X_{p-1}) \arrow[r] \arrow[d] & H_n(X_{p-1}, X_{p-2}) \arrow[r] & H_{n-1}(X_{p-2}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X_{p+1}) \arrow[r] \arrow[d] & H_{n+1}(X_{p+1}, X_p) \arrow[r] & H_n(X_p) \arrow[r] \arrow[d] & H_n(X_p, X_{p-1}) \arrow[r] & H_{n-1}(X_{p-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X_{p+2}) \arrow[r] \arrow[d] & H_{n+1}(X_{p+2}, X_{p+1}) \arrow[r] & H_n(X_{p+1}) \arrow[r] \arrow[d] & H_n(X_{p+1}, X_p) \arrow[r] & H_{n-1}(X_p) \arrow[r] \arrow[d] & \cdots \\
& \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

And we pack this staircase up into an exact couple with $n = p+q$.

Hyperhomology

Def.: Let $\mathcal{A}$ be an abelian category with enough projectives. A (left) Cartan-Eilenberg resolution $P_{*,*}$ of a chain complex $A_* \in Ch(\mathcal{A})$ is an upper half-plane double complex of projectives together with an augmentation $\varepsilon: P_{*,0} \to A_*$ such that for every p,

① $A_p = 0 \Rightarrow P_{p,*} = 0 \quad \forall *$

② $B_p(\varepsilon): B_p(P, d^h) \to B_p(A)$ (boundaries)
$H_p(\varepsilon): H_p(P, d^h) \to H_p(A)$
are projective resolutions in $\mathcal{A}$.

Lemma: Every chain complex has a Cartan-Eilenberg resolution.

Def.: Let $f, g: D \to E$ be two maps of double complexes. A chain homotopy from f to g consists of maps $s^h_{p,q}: D_{p,q} \to E_{p+1,q}$ and $s^v_{p,q}: D_{p,q} \to E_{p,q+1}$ so that

$$g - f = (d^h s^h + s^h d^h) + (d^v s^v + s^v d^v)$$
$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0.$$

$\rightsquigarrow$ now $s^h + s^v: Tot(D)_n \to Tot(E)_{n+1}$ forms a chain homotopy.

Def.: Let $F: \mathcal{A} \to \mathcal{B}$ be right exact and assume, that $\mathcal{A}$ has enough projectives. Let $A \in Ch(\mathcal{A})$ and $P_{\cdot,\cdot} \to A$ a Cartan-Eilenberg resolution. Define $\mathbb{L}_i F(A)$ to be $H_i\, Tot^{\oplus}(F(P_{\cdot,\cdot}))$.

We get a functor this way $\mathbb{H}_i F : Ch(\mathcal{A}) \to \mathcal{B}$ called the i^{th} left hyper-derived functor of F. (Assuming $\mathcal{B}$ is cocomplete.)

We may dually define the right hyper-derived functor of a left exact functor.

We get the following spectral sequences arising from the double cochain complex $F(I)$, where $A_\cdot \to I_{\cdot,\cdot}$ is an injective resolution of $A_\cdot \in Ch_{\geq 0}(\mathcal{A}^{op})$ (we need boundedness)

(weakly convergent) $\quad {}^{II}E_2^{p,q} \cong (R^pF)(H^q(A)) \Rightarrow \mathbb{R}^{p+q}F(A) = H^{p+q}(\mathrm{Tot}^{\pi}(F(A)))$

$$ {}^{I}E_2^{p,q} \cong H^p(R^qF(A)) \Rightarrow \mathbb{R}^{p+q}F(A).$$

Grothendieck spectral sequences

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be left exact functors.

Def: Let $F : \mathcal{B} \to \mathcal{C}$ be a left-exact functor. $B \in \mathcal{B}$ is called F-acyclic if $\forall i \neq 0 \;\; R^iF(B) = 0$.

Theorem: In the above setup, suppose, that G sends injectives to F-acyclic objects. Under this assumption, there is a convergent first quadrant cohomological spectral sequence for all $A \in \mathcal{A}$:

$${}^{I}E_2^{p,q} \cong (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

Proof:

Choose an injective resolution $A \to I$ of A in $\mathcal{A}$, apply G to get $G(I)$ a cochain complex in $\mathcal{B}$.

Using a first quadrant Cartan-Eilenberg resolution of $G(I)$, form the hyper derived functors $\mathbb{R}^nFG(I)$. Now we get two spectral sequences:

The first: $${}^{I}E_2^{p,q} = H^p(\underbrace{R^qF(G(I))}_{=0\ \forall q \neq 0}) \Rightarrow (\mathbb{R}^{p+q}F)(G(I))$$

$$\Rightarrow (\mathbb{R}^pF)(GI) \cong H^p(FG(I)) = R^p(FG)(A).$$

The second: $${}^{II}E_2^{p,q} = (R^pF)(H^q(G(I))) \Rightarrow (\mathbb{R}^{p+q}F)(G(I)) = R^p(FG)(A)$$

Since $H^q(G(I)) = R^pG(A)$ we get the Grothendieck s. seq. $\square$

The Leray spectral sequence

Let $f: X \to Y$ be a map of topological spaces. Let $Sh(X) = \{F : X_{Zar}^{op} \to Ab \mid F \text{ is a sheaf}\}$.

$$\begin{tikzcd} Sh(X) \arrow[r, "f_*"] \arrow[d, hook] & Sh(Y) \arrow[d, hook] \\ PSh(X) = Ab^{X_{Zar}^{op}} \arrow[r, "f_*"] \arrow[ur, phantom, "\circlearrowleft"] & Ab^{Y_{Zar}^{op}} = PSh(Y) \end{tikzcd}$$

$f : Y_{Zar} \to X_{Zar}$

$\underset{\text{open}}{V \subseteq Y} \mapsto \underset{\text{open}}{f^{-1}(V) \subseteq X}$

sheafification is left adjoint, so left Kan extension along f composed with sheafification provides a left adjoint

$$\begin{tikzcd} f^{-1}: Sh(Y) \arrow[r] \arrow[d] & Sh(X) \\ PSh(Y) \arrow[r, "Lan_f"'] & PSh(X) \arrow[u, "ass"'] \end{tikzcd}$$

f^{-1} is exact: the pointwise Kan-extension formula which provides it is a filtered colimit hence exact, see Theorem 2.6.15.

Thus f_* is left exact & preserves injectives.

Let $\mathcal{F} \in Sh(X)$, we have

$$\begin{tikzcd} Sh(X) \arrow[rr, "f_*"] \arrow[dr, "\Gamma"'] & & Sh(Y) \arrow[dl, "\Gamma"] \\ & Ab & \end{tikzcd}$$

The Grothendieck spectral sequence now yields the Leray spectral sequence:

$$E_2^{p,q} = H^p(Y; R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X; \mathcal{F})$$

where recall that $H^q(Y; \mathcal{G}) = R^q\Gamma(\mathcal{G})$.