Spectral sequences

Def.: A homological spectral sequence (starting on the E^a-page) in an abelian category $\mathcal{A}$ consists of the following:

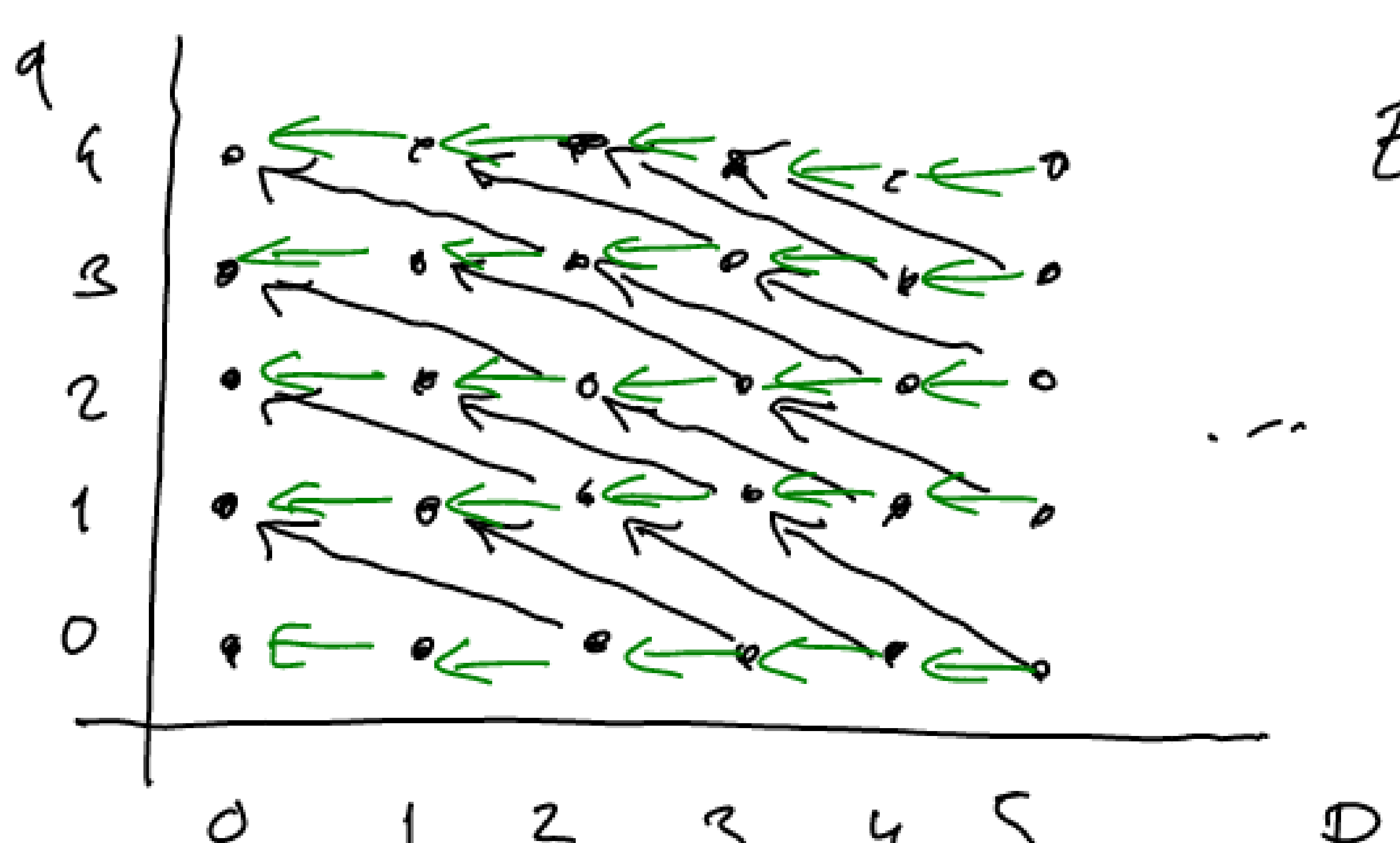
① A family $\{E^r_{p,q}\}^{a \leq r}_{p,q \in \mathbb{Z}}$ of objects in $\mathcal{A}$

② Maps $\{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}^r_{p,q}$ called differentials, s.t. $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$.

③ Isomorphisms $E^{r+1}_{p,q} \cong \dfrac{\ker(d^r_{p,q})}{\operatorname{im}(d^r_{p+r,q-r+1})}$.

Remark: A priori no connection is given between d^r & d^{r+1}.

Picture:



E^2-page

E^1-page

first quadrant example

no hitting points above us in the next diagonal!

Maps of spectral sequences are defined in as bigraded maps of degree $(0,0)$ on each page that commute with the differentials and that are so that the map from the previous page induces the map on the next page.

Def.: A spectral sequence $\{E^r_{p,q}\}$ converges to H_*, if

① there is a (*) filtration of H_* whose associated graded is the E^∞-page, i.e. a sequence of inclusion of subobjects of H_*,

$$\cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq H_n \qquad \forall n \in \mathbb{Z}$$

with $\underbrace{\bigcap_p F_p H_n = 0}_{\text{Hausdorff}}$ and $\underbrace{\bigcup_p F_p H_n = H_n}_{\text{exhaustive}}$, such that

$$E^\infty_{p,q} := \operatorname*{colim}_r E^r_{p,q} \cong F_p H_{p+q} / F_{p-1} H_{p+q} ,$$

and

② $\forall p,q \ \exists r_0 \ \forall r > r_0 \quad d^r_{p,q} = 0$. (the spectral sequence is regular)

and

③ $H_n = \lim_p (H_n / F_p H_n) \ \forall n$.

<u>Remark</u>: Assume the spectral sequence is first quadrant, i.e.

$E^r_{p,q} = 0$ whenever $p<0$ or $q<0$,

then ② is automatic, $E^\infty_{p,q} \cong E^r_{p,q}$ for large enough r,
the filtration must be finite as $0 = F_{-1} H_n \subseteq F_0 H_n \subseteq \cdots \subseteq F_n H_n = H_n$ (✱)
so ③ is automatic from ① so the assumption becomes that
there is a filtration as (✱) with associated graded $\bigoplus_p E^\infty_{p,n-p}$.

<u>Remark</u>: To recover H_* from a spectral sequence $\{E^r_{p,q}\}$ with $E^r_{p,q} \Rightarrow H_*$ is not necessarily possible: it involves extension problems: $F_p H_n$ will have to be an extension

$$0 \to F_{p-1} H_n \longrightarrow F_p H_n \to E^\infty_{p,n-p} \to 0$$

<u>Def.</u>: Let $E^r_{p,q} \Rightarrow H_*$, $E'^r_{p,q} \Rightarrow H'_*$ suppose that we have maps $f : E^r_{p,q} \to E'^r_{p,q}$ and $h : H_* \to H'_*$ we say that f is compatible with f, in case $h(F_p H_n) \subseteq F_p H'_n$ $\forall p,n$ and

$$\begin{tikzcd}
F_p H'_n \arrow[r, two heads] & F_p H'_n / F_{p-1} H'_n \arrow[r, phantom, "\cong"] & E'^\infty_{p,n-p} \\
F_p H_n \arrow[u, "h"] \arrow[r, two heads] & F_p H_n / F_{p-1} H_n \arrow[u, phantom, "\circlearrowleft"] \arrow[r, phantom, "\cong"] & E^\infty_{p,n-p} \arrow[u, "f^\infty"']
\end{tikzcd}$$

Theorem: In the setup of the above definition, if

$\forall p,q \ \exists r$ s.t. $f^r : E^r_{p,q} \to E'^r_{p,q}$ is an iso. (hence f^∞ is an iso),

then $h : H_* \to H'_*$ is an isomorphism.

$\rightsquigarrow$ an application of the 5-lemma

First example: The Leray-Serre spectral sequence.

Def.: Let $p : X \longrightarrow B$ be a Serre fibration, that is a map that satisfies the right lifting property with respect to any map in $\{D^n \times \{0\} \hookrightarrow D^n \times [0,1]\}_{n \geq 0}$:

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Remark: Any fiber bundle (so any covering) is a Serre fibration.

Def.: A fiber sequence is a sequence of maps of topological spaces

$F \xrightarrow{i} X \xrightarrow{p} B$ s.t.

fiber total space base space

① i is an injection

② $i(F) = p^{-1}(b_0)$ for some $b_0 \in B$

③ p is a Serre fibration.

Theorem: Given any fiber sequence $F \xrightarrow{i} X \xrightarrow{p} B$ there is a naturally constructed long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_k(F, x_0) \xrightarrow{i_*} \pi_k(X, x_0) \xrightarrow{p_*} \pi_k(B, b_0) \xrightarrow{\partial} \pi_{k-1}(F, x_0) \longrightarrow \cdots$$

Theorem (Leray-Serre spectral sequence): Let $F \xrightarrow{i} X \xrightarrow{p} B$ be a Serre fibration such that B is simply connected. There is a natural first quadrant spectral sequence starting on the E^2-page such that

$$E^2_{p,q} \cong H_p(B; H_q(F)) \Rightarrow H_{p+q}(X).$$

Natural in the sense that a map of fiber sequences

$$\begin{array}{ccccc} F & \xrightarrow{i} & X & \xrightarrow{p} & B \\ \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta \\ F' & \xrightarrow{i'} & X' & \xrightarrow{p'} & B' \end{array}$$

induces a map $f: E^{\cdot}_{\cdot,\cdot} \to E'^{\cdot}_{\cdot,\cdot}$ of spectral sequences compatible with $\alpha_*: H_*(X) \to H_*(X')$, and such that $f^2_{p,q} = (\beta_*, \gamma_*): H_p(B; H_q(F)) \longrightarrow H_p(B'; H_q(F'))$.

Example:

① Let us compute $H_*(\mathbb{R}P^3)$: $\rightsquigarrow$ I am omitting coefficients, this works over any

We have the Hopf map $S^3 \xrightarrow{\eta} S^2$

$$S^3 = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\} \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{C}P^1 \cong S^2$$

$$(x,y) \longmapsto (2x\bar{y}, |x|^2 - |y|^2)$$

$$S^2 = \{(z,x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1\}$$

$$4|x|^2|y|^2 + |x|^4 - 2|x|^2|y|^2 + |y|^4 = 1^2 = 1$$

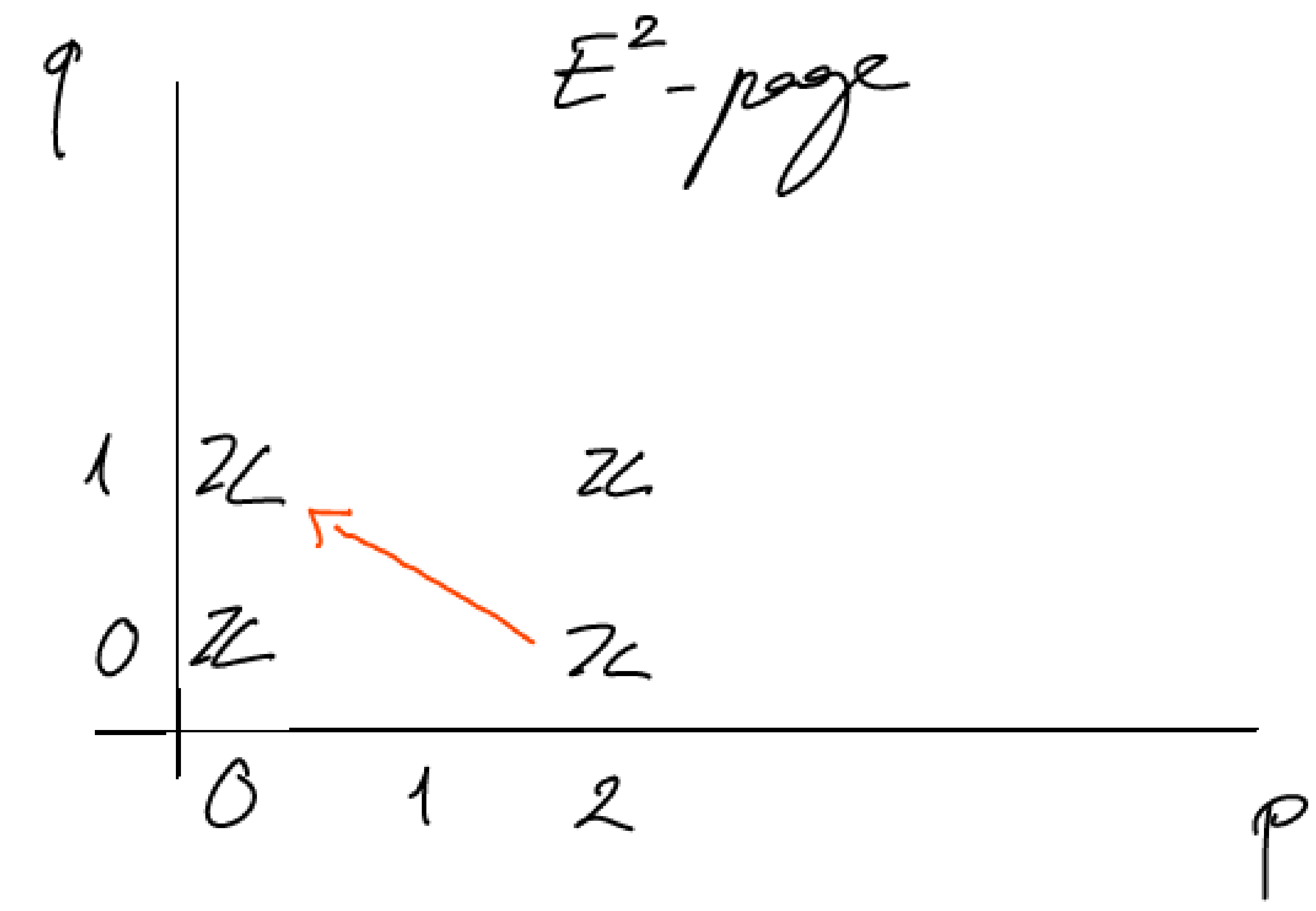
this is a fibration with fiber S^1

furthermore we can consider

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\ \downarrow \cdot 2 & & \downarrow & & \| \\ S^1 & \longrightarrow & \mathbb{R}P^3 & \dashrightarrow & S^2 \end{array}$$

since $\eta(-x,-y) = \eta(x,y)$.

Now let us inspect the spectral sequence associated to $S^1 \to \mathbb{R}P^3 \xrightarrow{\eta} S^2$.

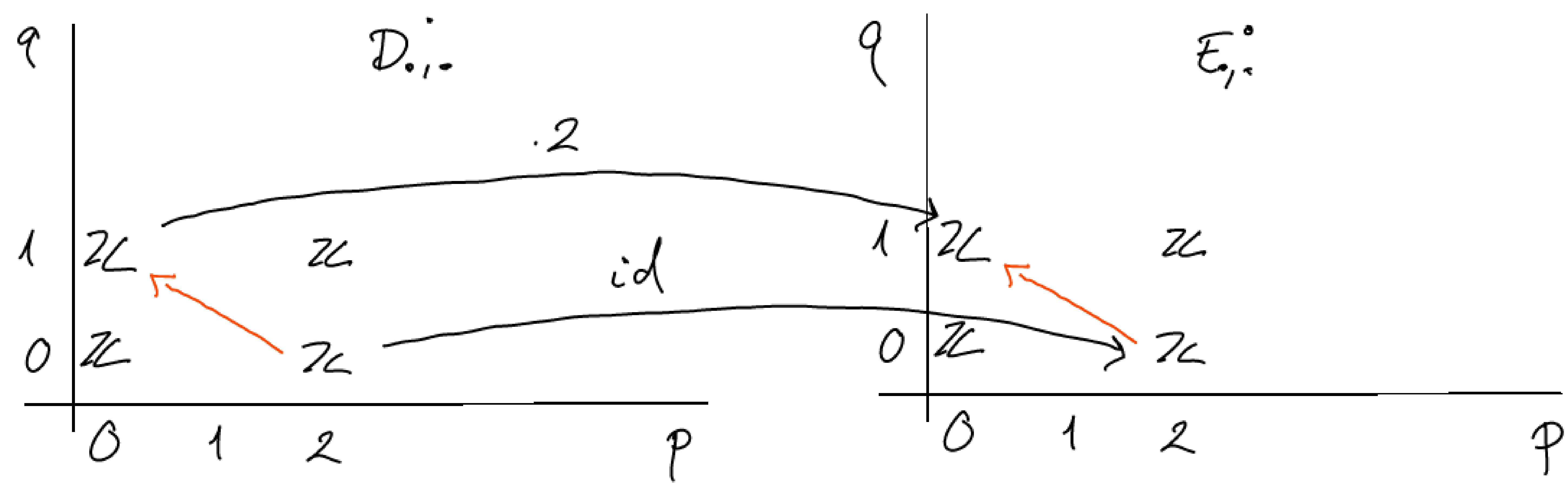


$H_p(S^2; H_q(S^1))$

the only potentially non-zero differential in the spectral sequence.

how can we figure out what this map should be?
We have a pretty good grasp on how the spectral sequence of $S^1 \to S^3 \xrightarrow{\eta} S^2$ should behave; it looks exactly the same, and we know the homology of S^3 and thus we can deduce that the red map has to be an isomorphism there.

$S_1 \to S^3 \to S^2$ $\qquad\qquad$ $S^1 \to \mathbb{R}P^3 \to S^2$



converges to $H_*(S^3) = \begin{cases} \mathbb{Z} & \text{if } * \in \{0,3\} \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow {}^{D}d^2_{2,0} : D^2_{2,0} \to D^2_{0,1}$
has to be injective to kill $D^2_{2,0}$
has to be surjective to kill $D^2_{0,1}$

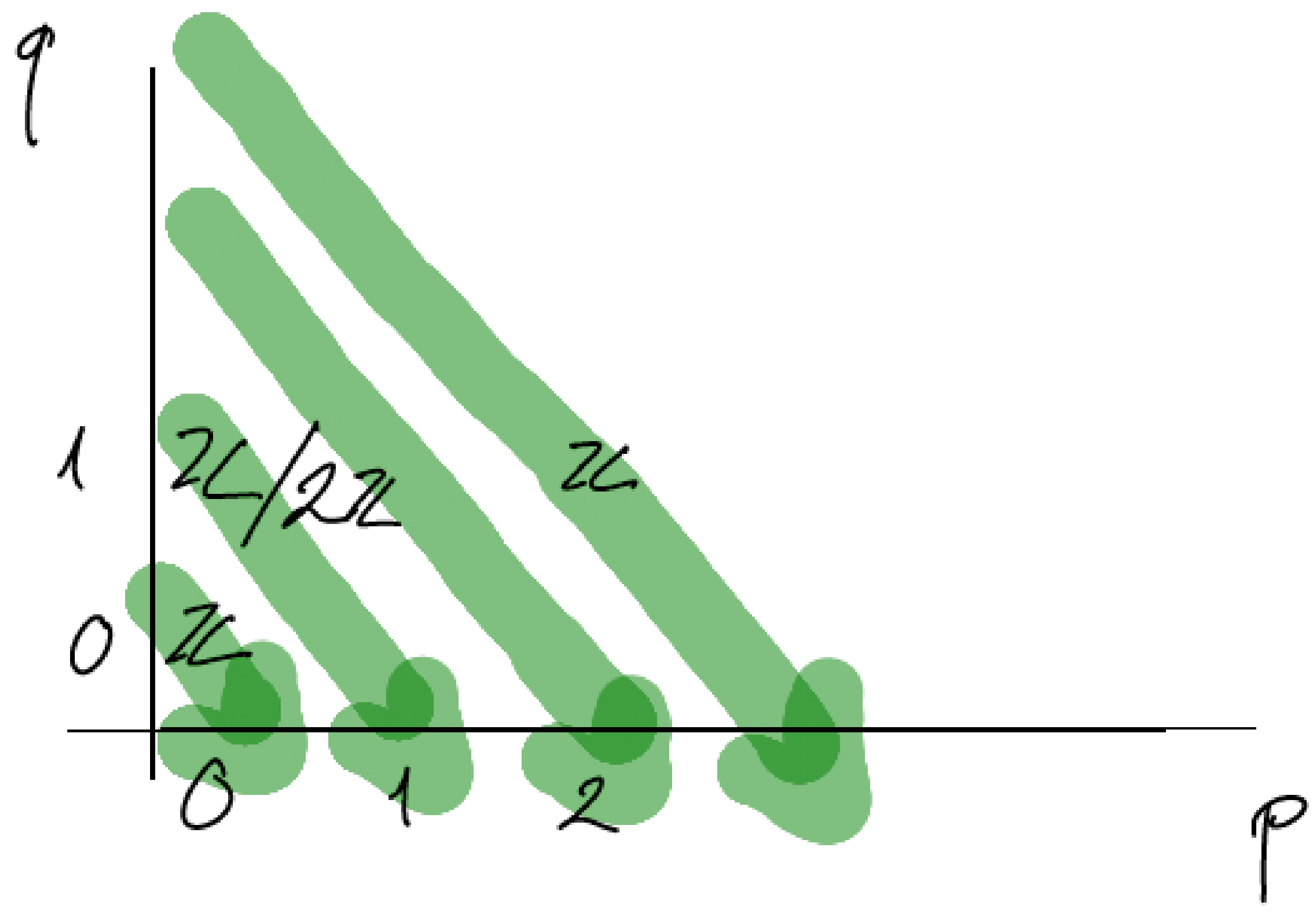
remember we have

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\cdot 2"] & S^3 \arrow[r, "\eta"] \arrow[d] & S^2 \arrow[d, equal] \\ S^1 \arrow[r] & \mathbb{R}P^3 \arrow[r] & S^2 \end{tikzcd}$$

$$\begin{tikzcd} H_0(S^2; H_1(S^1)) \arrow[r, "\cdot 2"] & H_0(S^2; H_1(S^2)) \\ H_2(S^2; H_0(S^1)) \arrow[u, "{}^{D}d^2_{2,0}"] \arrow[r, equal] \arrow[ur, phantom, "\circlearrowleft"] & H_2(S^2, H_0(S^1)) \arrow[u, "{}^{E}d^2_{2,0}"'] \end{tikzcd}$$

$\Rightarrow {}^{E}d^2_{2,0}$ is either the $\cdot 2$ or $\cdot(-2)$ map the E^3-page looks as follows: $E^3_{2,0} \cong \dfrac{\ker(d^2_{2,0})}{\operatorname{im}(d^2_{4,-1})} = \dfrac{\ker(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})}{0} = 0$

$$E^3_{0,1} \cong \frac{\ker(d^2_{0,1})}{\operatorname{im}(d^2_{2,0})} = \frac{\ker(\mathbb{Z} \to 0)}{\operatorname{im}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})} = \mathbb{Z}/2\mathbb{Z}$$



now there are no longer any non-zero differentials on any page due to degree reasons

thus $E^3_{\cdot,\cdot} \cong E^\infty_{\cdot,\cdot}$ and we say that the spectral sequence collapses.

let us read off the homology of $\mathbb{R}P^3$:

There is a filtration

$$\begin{tikzcd}
0 = F_{-1}H_1(\mathbb{R}P^3) \arrow[r, hook] & F_0 H_1(\mathbb{R}P^3) \arrow[r, hook] \arrow[d, two heads] & F_1 H_1(\mathbb{R}P^3) = H_1(\mathbb{R}P^3) \arrow[d, two heads] \\
 & E^\infty_{0,1} \cong F_0H_1(\mathbb{R}P^3)/F_{-1}H_1(\mathbb{R}P^3) & E^\infty_{1,0} \cong F_1H_1(\mathbb{R}P^3)/F_0H_1(\mathbb{R}P^3)
\end{tikzcd}$$

the associated graded is the anti-diagonal in the E^∞-page

$E^\infty_{0,1} = \mathbb{Z}/2\mathbb{Z}$ so $F_0 H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$0 \to \mathbb{Z}/2\mathbb{Z} \to F_1H_1(\mathbb{R}P^3) \to \underbrace{E^\infty_{1,0}}_{0} \to 0$$

$$\Rightarrow H_1(\mathbb{R}P^3) \cong F_1H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2\mathbb{Z}.$$

similarly

$$H_*(\mathbb{R}P^3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } * \in \{0,3\} \\ \mathbb{Z}/2 & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

Cohomological spectral sequences

⇝ the same but for cohomology

<u>Def.</u>: A cohomological spectral sequence (starting on the E_a-page) in an abelian category $\mathcal{A}$ consists of the following:

① A family $\{E_r^{p,q}\}_{a \leq r}^{p,q}$ of objects in $\mathcal{A}$

② Maps $\{d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\}_{p,q}^r$ called differentials, s.t.
$d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$. ⇝ total degree +1

③ Isomorphisms $E_{r+1}^{p,q} \cong \dfrac{\ker(d_r^{p,q})}{\mathrm{im}(d_r^{p-r,q+r-1})}$.

Let me just state bounded convergence in this case:

⇝ a spectral sequence is bounded if for each n there is only finitely many non-zero terms of total degree n in $E^a_{*,*}$.

bounded ⇒ regular

<u>Def.</u>: $E_r^{p,q} \Rightarrow H^*$ if there is a finite filtration
$$0 = F^t H^n \subseteq \dots \subseteq F^{p+1} H^n \subseteq F^p H^n \subseteq \dots \subseteq F^s H^n = H^n \text{ s.t.}$$
$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$.

There is unsurprisingly a cohomological version of the Leray-Serre spectral sequence. Recall that there is a ring structure on the cohomology of a topological space given by the cup product.

So how does the cup product influence the spectral sequence?

Suppose that we have some multiplication map

$$\cdot : E_r^{p_1,q_1} \otimes E_r^{p_2,q_2} \longrightarrow E_r^{p_1+p_2,\, q_1+q_2}$$ such that

it defines a bigraded commutative ring: $x \cdot y = (-1)^{|x||y|} y \cdot x$

where $x \in E_r^{p_1,q_1}$, $y \in E_r^{p_2,q_2}$ $|x| = p_1+q_1$, $|y| = p_2+q_2$

furthermore assume, that d^r satisfies the Leibnitz relation

$$d^r(x \cdot y) = d^r(x) \cdot y + (-1)^{|x|} x \cdot d^r(y).$$

This means that the multiplication induces a multiplication on $E_{r+1}^{\cdot,\cdot}$, we assume this will be the multiplication on E_{r+1}:

$$[x][y] = [xy], \quad [d^r(x)][y] = [d^r(x)y] = [d^r(xy)]$$

$$\text{as } d^r(x,y) = d^r(x)y + (-1)^{|x|} x \cdot \underbrace{d^r(y)}_{0}$$

$$\text{as } y \in \ker(d^r).$$

These conditions so far ensure, that the multiplication works well with the spectral sequence. This is already useful for deducing differentials (provided we understand the multiplicative structure), but we can also specify what we can expect from the behavior of the multiplicative structure with respect to convergence.

Suppose that H^* is a graded commutative ring and $E_r^{p,q} \Rightarrow H^*$, assume that the filtration exhibiting this is multiplicative in the sense, that $(F^pH^*) \cdot (F^qH^*) \subseteq F^{p+q}H^*$. We ask in this case for

$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$ to be compatible with the ring structure.

The cohomological Leray-Serre spectral sequence will satisfy these assumptions and this compatibility.

<u>Theorem:</u> Let $F \to X \to B$ be a fiber sequence. If B is simply connected, then there is a natural spectral sequence

$$E_2^{p,q} \cong H^p(B; H^q(F)) \Rightarrow H^{p+q}(X).$$

Furthermore, the composition of the external cup product with the cup product on the coefficients

$$H^{p_1}(B; H^{q_1}(F)) \otimes H^{p_2}(B; H^{q_2}(F)) \to H^{p_1+p_2}(B; H^{q_1}(F) \otimes H^{q_2}(F)) \longrightarrow H^{p_1+p_2}(B; H^{q_1+q_2}(F))$$

defines a bigraded ring structure on $E_2^{\bullet,\bullet}$ with respect to which the differentials satisfy the Leibnitz rule and the convergence is compatible with the graded ring structure of $H^*(X)$.

<u>Remark:</u> In sufficiently nice cases (certainly when we work over a field) the Künneth formula provides

$$H^p(B; H^q(F)) \cong H^p(B) \otimes H^q(F).$$

In this case, the ring structure is $(x \otimes y) \cdot (x' \otimes y') = (-1)^{|y||x'|} (x \cup x') \otimes (y \cup y')$.

<u>Example:</u> Calculating the cohomology ring of $\mathbb{C}P^\infty$.

There is a fiber sequence $S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$.

One can derive this by hand. There are multiple ways of thinking

about this: it is the special case of $S^1 \to ES^1 \to BS^1$,
it can be thought of as $\underbrace{\Omega K(\mathbb{Z},2)}_{\substack{\cong \\ K(\mathbb{Z},1)}} \to P_0 K(\mathbb{Z},2) \xrightarrow{ev_1} K(\mathbb{Z},2)$

or the delooping of $\mathbb{Z} \to \mathbb{R} \xrightarrow{\phi} S^1$, that is
$B\mathbb{Z} \to B\mathbb{R} \to BS^1$. ϕ is a $\mathbb{Z}$-principal bundle

In any case, we get a spectral sequence

$$\begin{array}{c|c}
q & \\
 & 0 \\
1 & \text{——} \; H^p(\mathbb{C}P^\infty;\mathbb{Z}) \; \text{——} \\
0 & \text{——} \; H^p(\mathbb{C}P^\infty;\mathbb{Z}) \; \text{——} \\
\hline
 & p
\end{array}$$

and being path connected we know $H^0(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

Now observe that $S^\infty \simeq *$ and therefore the spectral sequence must converge to zero.

this column must be zero as no non-zero differential has it as source or target

$$\begin{array}{c|ccccccc}
1 & \mathbb{Z} & 0 & H^2(\mathbb{C}P^\infty) & 0 & H^4(\mathbb{C}P^\infty) & & \\
0 & \mathbb{Z} & 0 & H^2(\mathbb{C}P^\infty) & 0 & H^4(\mathbb{C}P^\infty) & & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & &
\end{array}$$

has to be an iso to kill both its source and target

we follow the inductive argument to get that the E^2-page is

$$\begin{array}{c|ccccccc}
q & & & & & & & \\
1 & \mathbb{Z} & & \mathbb{Z}c_1 & & \mathbb{Z}c_2 & & \\
0 & \mathbb{Z} & & \mathbb{Z}c_1 & & \mathbb{Z}c_2 & & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots & p
\end{array}$$

assume, that
$c_i \in H^{2i}(\mathbb{C}P^\infty) \cong \mathbb{Z}$ generates
$\forall i \geq 1$. WLOG we can assume
$d^2(c_i) = c_{i+1}$

from the Leibnitz rule we get

$$c_2 = d^2(c_1) = d^2(\overset{E^2_{2,0}\,\ni}{c_1} \cdot \overset{E^2_{0,1}\,\ni}{1}) = \underbrace{d^2(c_1)}_{\substack{0 \\ \text{as it is in } E^2_{4,-1}}} \cdot 1 + c_1 \cdot \underbrace{d^2(1)}_{c_1} = c_1^2$$

$$c_3 = d^2(c_2) = d^2(\overset{E^2_{2,0}\,\ni}{c_1} \cdot \overset{E^2_{2,1}\,\ni}{c_1}) = \underbrace{d^2(c_1)}_{0} \cdot c_1 + c_1 d^2(c_1) = c_1 d^2(c_1) = c_1 c_2 = c_1^3$$

$\vdots$ induction step:

$$c_{i+1} = d^2(c_i) = d^2(c_1 \cdot c_1^{i-1}) = c_1 d^2(c_1^{i-1}) = c_1 c_i = c_1^{i+1}.$$

Therefore we obtain $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[c]$ with $|c| = 2$.