Reading Group Talk 13/05/2026

Del Pezzo Surfaces

Recall the classification of Algebraic Curves:

genus	0	1	≥ 2
canonical sheaf ω_X	anti-ample	trivial ($\mathcal{O}_X$)	ample

Similarly, we may classify higher-dimensional varieties by how ample ω_X is.

§ Kodaira Dimension

NB// Throughout, define a "nice" variety to be smooth, projective and geometrically integral.

Let X be a nice variety over a field k.

We associate to X an element

$$\kappa = \kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$$

called the Kodaira dimension of X.

Let ω_X be the canonical sheaf

Case 1:

$$H^0(X, \omega_X^{\otimes m}) = 0 \ \forall m \in \mathbb{Z}_{\geq 1}.$$

global sections

Then define $\kappa(X) = -\infty$

Case 2:

$$H^0(X, \omega_X^{\otimes m}) \neq 0 \text{ for some } m \in \mathbb{Z}_{\geq 1}$$

If m is s.t. $H^0(X, \omega_X^{\otimes m}) \neq 0$ then a choice of basis of global sections defines a rational map

$$\phi_m : X \dashrightarrow \mathbb{P}^N$$

for some N depending on m.

ϕ_m is defined on an open subscheme U_m of points at which the global sections generate $\omega_X^{\otimes m}$

Let $\overline{\phi_m(X)}$ be the Zariski closure of $\phi_m(U_m)$ in $\mathbb{P}^N$.

Then for $m > 1$ sufficiently large and divisible, $\overline{\phi_m(X)}$ is independent of m up to birat. equiv.

We let $K(X)$ be its dimension. i.e.

$K := \max_m \dim \overline{\phi_m(X)}$.

e.g. (curves)

Let X be a nice genus g curve. Then:

$g=0 \Rightarrow K(X) = -\infty$
$g=1 \Rightarrow K(X) = 0$
$g=2 \Rightarrow K(X) = 1$

Prop 9.12.
Kodaira dimension is ~~stable under~~ invariant under birational equivalence and base-extension.

can skip

Def^n
If $K(X) = \dim(X)$ (The maximum possible) then call X general type

e.g.
If ω_X is ample then X is of general type.

Warning! The converse is not true in general.

§ Notions of Rationality

Def^n
X an n-dim integral variety over alg. closed fld. K.

- X rational if X is birational to $\mathbb{P}^n$
- X stably rational if $\exists\, m \in \mathbb{N}$ s.t. $X \times \mathbb{P}^m$ is rational
- X unirational if $\exists$ dominant rational map $\mathbb{P}^N \dashrightarrow X$ for some $N \geq 0$.

All of the above are examples of $K(X) = -\infty$.

An important notion in the context of rational points is the notion of rationally connected.

Say out loud:
Vaguely: Rationally connected $= \exists$ a family of rational curves s.t. for almost every pair of points on the variety, there is a curve in the family joining them.

Def^n A rational curve in X is a rational map $f: \mathbb{P}^1 \dashrightarrow X$ (Equivalently, a morphism)

We say that $x, x' \in X(K)$ are joined by a rational curve if there is a rat. curve f such that $x, x' \in f(\mathbb{P}^1)$

An Algebraic family of rational curves parametrised by a base variety B is a rational map $F : B \times \mathbb{P}^1 \dashrightarrow X$

This is a family in the sense that for (almost) every $b \in B(K)$ the restriction of F to $\{b\} \times \mathbb{P}^1$ is a rational map $\mathbb{P}^1 \dashrightarrow X$.

Given such a family, the pairs of points that it joins are the pairs of the form $(F(b,t), F(b,t'))$ for some $b \in B(K)$, $t, t' \in \mathbb{P}^1(K)$.

Def^n

The variety X is rationally connected if there is a variety B and rational map $F : B \times \mathbb{P}^1 \dashrightarrow X$ s.t. the rational map

$$B \times \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X \times X$$
$$(b, t, t') \longmapsto (F(b,t), F(b,t'))$$

is dominant.

i.e. a dense subset of pairs (x, x') in $X \times X$ are

connected by a rational curve:

Prop:

If X is rat. connected then any general pair of points can be joined by a rational curve:

i.e. there is a dense open subset U of $X \times X$ s.t. any pair $(x, x') \in U(K)$ can be joined by a rational curve. The converse holds if K is uncountable.

§ Fano Varieties

Let X be a nice variety over a field K, ω_X its canonical sheaf.

Def^n

X is Fano if $\omega_X^{\otimes(-1)}$ is ample

e.g. If $X = \mathbb{P}^n$ then $\omega_X^{\otimes(-1)} \cong \mathcal{O}(n+1)$ so X is Fano.

e.g. If X is a nice curve, a line bundle on X is ample iff $\deg \mathcal{L} > 0$, and $\deg(\omega_X^{\otimes(-1)}) = 2 - 2g > 0$ iff $g = 0$.

e.g.

A nice hypersurface of degree d in $\mathbb{P}^n$ is Fano iff $d \leq n$

Prop:

Fano is invariant under base extension.

Some Implications

$$\text{Fano}$$
$$\Downarrow$$
$$\text{Rational} \Rightarrow \text{Stably Rational} \Rightarrow \text{Unirational} \Rightarrow \text{Rationally Connected}$$
$$\Downarrow$$
$$\kappa = -\infty$$

Let X be Fano. Let $m = \dim X$. Let K be a canonical divisor on X.

If $-K$ is very ample then the complete linear system $|-K|$ embeds X as a subvariety of some $\mathbb{P}^n$.

The degree of the subvariety is the number of points (counted with multiplicity) resulting from cutting

it with m general hyperplanes in $\mathbb{P}^n$.

Equivalently, the degree equals the self intersection number of $(-K)^m$ on X.

In general, $-K$ is only ample, but some positive multiple is very ample, so $(-K)^m$ is still positive, so define the degree of X to be $(-K)^m$.

Warning!

Fano is not invariant under birational maps!

Non-Closed ground fields

Let k be non-algebraically closed.

X a nice k-variety.

X is called adjective if $X_{\bar{k}}$ is adjective.

If a property holds over $\bar{k}$ we use a prefix $\bar{k}$-

Conjecture (Colliot-Thélène)

Let X be nice variety / # fld. k.

Suppose X is rationally connected. Then the Brauer-Manin obstruction is the only obstruction to the local-global principle.

§ Del Pezzo Surfaces: Generalities

Defn
A del Pezzo Surface is a (nice) Fano Variety of dimension 2.

According to the general defn for Fanos, the degree of a dP surface X is a positive integer $d := (-K)\cdot(-K) = K\cdot K$

It turns out that $\dim H^0(X, -K) = d+1$ and that $-K$ is very ample for $d \geq 3$.

Thus if $d \geq 3$, $|-K|$ embeds X as a degree d surface in $\mathbb{P}^d$.

Shorthand: "del Pezzo surface of degree n" = "dPn" e.g. dP3 or dP6

Over Separably closed Fields: Classification

Defn Let $0 \leq r \leq 8$. Points $P_1, \dots, P_r \in \mathbb{P}^2(k)$ are in general position if they are distinct and none of the following conditions hold:

i) 3 P_i lie on a line
ii) 6 ——— conic
iii) 8 ——— Singular cubic with 1 P_i at the singularity.

Theorem (Classification of dP Surfaces)

Let k be a separably closed field.
Let X be a dP surface over k.

Then exactly one of the following holds:
- $X \cong \mathbb{P}^1 \times \mathbb{P}^1$; then $\deg X = 8$
- $\exists r$ with $0 \leq r \leq 8$ s.t. X is the blowup of r k-points in general position. Then $\deg X = 9 - r$.

Notably: dP surfaces are rational!

Remark
If X is as above, then $\operatorname{Pic} X \cong \mathbb{Z}^{10-d}$:

This is true automatically for $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, and blowing up a k-point adds a factor of $\mathbb{Z}$ corresponding to the divisor class generated by the class of the exceptional curve.

One can also explicitly describe the canonical class and the intersection pairing on $\operatorname{Pic} X$ w.r.t. a suitable basis.

Remark

Let $X = Bl_{p_1,\dots,p_r}\mathbb{P}^2$ for $r \geq 9$.

Then $K \cdot K = 9 - r \leq 0$, so X is not Fano, so not dP.

$\Rightarrow$ Being dP is not preserved under Birat. maps.

Over an arbitrary field

~~Theorem~~ ~~(9.4.8. in Poonen)~~

"Recall" Prop 1.5.25 in Poonen which gives some equivalent defns of "$\dim k \leq 1$" fields:

defn

Say a field k has $\dim k \leq 1$ if:

$Br\, L = 0 \ \forall$ alg. extension L/k.

e.g. finite fields, an extension of alg. closed field of transcendence degree 1, the maximal unramified extension of a local field with perfect residue field.

Theorem (9.4.8 in Poonen)

Let k be a field, X dP over k of degree $d \geq 5$:

i) If $d = 7$ or 5, then X has a k-point

ii) If $\dim k \leq 1$ then X has a k-point

iii) If X has a k-point then X birational to $\mathbb{P}^2_k$

iv) If X has a k-point and k is infinite, then $X(k)$ is Zariski dense in X

v) The homomorphism $Br(k) \to Br(X)$ is surjective

vi) If k is a global field, then X satisfies local-global principle

vii) If ________ weak approximation.

Remark:

When $d \leq 4$ most parts can fail.

§ Some selected degree-by-degree results

Degree 9

$X_{k_s} \cong \mathbb{P}^2_{k_s}$, so X is a Severi-Brauer surface.

In particular, recall from my previous talk:

- If X has a k-point, $X \cong \mathbb{P}^2_k$
- $\dim k \leq 1 \Rightarrow X \cong \mathbb{P}^2_k$
- If k is a global field, X satisfies local-global.

Degree 8

Let X be dP8 over k. Then exactly one of the following holds:

1) There is a degree 2 étale ext. $L \supseteq k$ and a nice conic C over L s.t. X is isomorphic to the restriction of scalars $\mathrm{Res}_{L/k}\, C$.

(In the split case $L = k \times k$, this just means that X is the product of 2 nice conics over k)

2) X is the blowup of $\mathbb{P}^2_k$ at a k-point.

Skip if short on time

Corollary:
If $\dim k \leq 1$, any degree 8 dP surface X over k has a k-point.

Proof: 2) is obvious
1) Since $\dim k \leq 1$, we have $\mathrm{Br}\, L = 0$
Since C is 1-dim Severi-Brauer variety over L, it has an L point
Finally, $X(k) = C(L)$. $\square$

Corollary: Let X be dP8 over k. If X has a k-point then X is birational to $\mathbb{P}^2_k$
2) is obvious.
1) If X has a k-point then C has an L-point.

So $C \cong \mathbb{P}^1_L$. $\mathbb{P}^1_L$ is birational to $\mathbb{A}^1_L$.

So X is birational to $\mathrm{Res}_{L/k}\, \mathbb{A}^1_L \cong \mathbb{A}^2_k$

$\mathbb{A}^2_k$ is birational to $\mathbb{P}^2_k$ $\square$.

Corollary
A dP8 over global k satisfies local-global.

Proof:
If $X = \mathrm{Res}_{L/k}\, C$, apply local-global to C, since conics satisfy local-global.

If $X = \mathrm{Bl}_p \mathbb{P}^2_k$ then X has a k-point already!

Degree 7

A dP7 is $\mathbb{P}^2_k$ blown up at either 2 k-points or at a closed point whose residue field is separable of degree 2 over k.

Degree 6

Let X be dP6 over k. If $\exists$ separable ext. K, L with $[K:k]=2$ and $[L:k]=3$ s.t. X has a K-point and an L-point, then X has a k-point.

Prop
Let X be dP6 over K. If either $\dim K \leq 1$ or K is global, and $X(\mathbb{A}) \neq \emptyset$, then X has a K-point.

Prop
Let X be dP6 over K. If X has a K-point then X is birational to $\mathbb{P}^2_K$.

Degree 5
Every dP5 has a K-point and is birational to $\mathbb{P}^2_K$

Now on to the lower degrees not covered by the main theorem...

Degree 4
These X are the intersection of 2 quadrics in $\mathbb{P}^4$.

If K is a global field, then local-global can fail!
Weak approximation can fail even if X has a K-point.

Degree 3
Perhaps some of the most well-studied examples. These are nice cubic surfaces ~~over~~ ~~K~~ in $\mathbb{P}^3$.

Mordell (falsely) conjectured that over $\mathbb{Q}$, these satisfy local-global.

Thm (Selmer):

$ax^3 + by^3 + cz^3 + dw^3 = 0 \subset \mathbb{P}^3_{\mathbb{Q}}$
$a, b, c, d \in \mathbb{Z} \setminus \{0\}$

Satisfies local-global if $ab = cd$ or $|abcd| \leq 500$.

Counterexample (Cassels, Guy)

$5x^3 + 9y^3 + 10z^3 + 12w^3 = 0$

violates local-global.

Also: W.A. can fail even if X is minimal and has a rat. point.

Degree 2

Now $\omega_X^{\otimes -1}$ is no longer very ample!

The anticanonical map here is a degree 2 morphism $X \to \mathbb{P}^2$ ramified along a curve of degree 4 in $\mathbb{P}^2$.

i.e. X is of degree 4 in a weighted projective space $\mathbb{P}(1,1,1,2)$

If k is a global field, then X may violate local-global

e.g. (Kresch - Tschinkel)

$$w^2 = -6x^4 - 3y^4 + 2z^4$$

[So here the ~~anticanonical~~ map is something like

$$[w:x:y:z] \mapsto [x:y:z]$$

which has degree 2 since you can also send

$$[-w:x:y:z] \mapsto [x:y:z]$$

The morphism ramifies on the curve

$$-6x^4 - 3y^4 + 2z^4 = 0 \subset \mathbb{P}^2$$

since here each point has only one pre-image.]

Also, Colliot-Thélène observed that one can obtain a counterexample by replacing z^2 by z^4 in Iskovskikh's surface (see Lewis' talk):

$$y^2 = -z^4 + (3w^2 - x^2)(x^2 - 2w^2)$$

Here, weak approx. can fail even if X is minimal and has a k-point.

Remark

So far, every example of a dP violating local-global can be explained by a BM obstruction.

Degree 1

X is dP1, is a degree 6 surface in WPS $\mathbb{P}(1,1,2,3)$

The common zero locus of any basis s_1, s_2 of the 2-dim space $H^0(X, -K)$ is independent of choice of basis.

This locus consists of a single degree 1 point since $(-K)\cdot(-K) = 1$.

i.e. the intersection of any 2 distinct divisors in $|-K|$ is a canonical k-point!

Thus $X(k) \neq \emptyset$.

There still might be a BM obs. to WA even if X is minimal.

See Poonen Section 9.4.12. for a table summarizing what is known

§ Quantitative Conjectures / Results

Manin's Conjecture (Source: Browning QAoPV)
Let S be a non-singular dP surface over $\mathbb{Q}$.
Let $\mathrm{Pic}(S) = \mathrm{Pic}_{\overline{\mathbb{Q}}}(S)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$
$\rho_S = \mathrm{rk}\,\mathrm{Pic}(S)$

Let $U \subset S$ be the Zariski open subset formed by removing from S the lines defined over $\overline{\mathbb{Q}}$.

Conjecture (Manin)
Let S be a degree d dP surface with $3 \leq d \leq 9$

Suppose $S(\mathbb{Q}) \neq \emptyset$. Then

$$N_U(B) = \#\{x \in U(\mathbb{Q}) : H(x) \leq B\} = C_S B (\log B)^{\rho_S + 1}$$
$$+ o(B (\log B)^{\rho_S+1})$$

Here, H is the anti-canonical height.

C_S : Conjectured by Peyre.

Degree 3

For non-singular cubic surfaces, the best known upper bound is
$$N_U(B) = O_{\varepsilon, S}(B^{4/3 + \varepsilon})$$
(Heath-Brown)

Which applies to any non-singular cubic surface S containing 3 coplanar lines over $\mathbb{Q}$.
e.g. Fermat cubic:
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$
Contains the lines:

- $x_1 + x_2 = x_3 + x_4 = 0$
- $x_1 + x_3 = x_2 + x_4 = 0$
- $x_1 + x_4 = x_2 + x_3 = 0$

lying in the plane
$$x_1 + x_2 + x_3 + x_4 = 0$$

The lower bound of the correct order of magnitude is known due to Sofos '14: For the semat cubic.

$$N_U(B) \gg B(\log B)^3$$

Manin's Conjecture has never been proven for any non-singular cubic!

It has been resolved for a number of ~~non~~ singular cubics.

Degree 4:

Manin's Conjecture proven for a non-singular $dP4$:

Theorem (Browning - de-la-Bretèche)

$S \subset \mathbb{P}^4$ given by

~~$Q_1(x_1, \ldots, x_5) := x_1x_2 - x_3x_4 = 0$~~

$$x_1x_2 - x_3x_4 = 0$$
$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - 2x_5^2 = 0$$

Then $N_U(B) = c_S B(\log B)^{\rho_S - 1}\left(1 + O\left(\frac{1}{\log\log B}\right)\right)$

where $\rho_S = 5$.

Heath-Brown's cubic bound can also be adapted to $dP4$

Degree ≥ 6

Non-singular $dP6$, $dP7$, $dP8$, $dP9$ are toric varieties. Manin's Conjecture proven for all toric varieties by Batyrev -- Tschinkel.

How often do we see B-M obstruction

This is a relatively new area of study, with a small amount of progress.

Q: Given a family of varieties, can you find an asymptotic formula for ~~[illegible]~~ how many up to height B have a Brauer-Manin obstruction?

Studied by:

K3 Surfaces
- Santens '23
- Gvirtz-Chen, Loughran, Nakahara '22
-

Châtelet Surfaces
- de-la-Bretèche, Browning '14
- Rome '19

dP surfaces:

- Jahnel, Schindler '16
- Mitankin, Salgado '22
- Shaw '24.

So far, not enough evidence to form a conjecture, but for example, the theorem of Harry Shaw for diagonal dP2 is as follows:

let $S_{\underline{a}}: a_0 x_0^4 + a_1 x_1^4 + a_2 x_2^4 = w^2$

$\subseteq \mathbb{P}_{\mathbb{Q}}(1,1,1,2)$

where $a_i \in \mathbb{Z} \setminus \{0\}$ be a family of diagonal dP2s

Theorem (Shaw '24)

$\exists$ a constant $A > 0$ s.t.

$$\#\left\{ \underline{a} \in (\mathbb{Z} \setminus \{0\})^3 : \begin{array}{l} |a_i| \leq T \;\forall i \in \{0,1,2\} \\ S_{\underline{a}} \text{ has BM obs. to} \\ \text{local-global} \end{array} \right\}$$

$$\sim A (T \log T)^{3/2}$$