2.4 Left derived functors

Use: To measure how far a functor is from being exact.

<u>Defn</u>: Let $F: \mathcal{A} \to \mathcal{B}$ be a right exact functor b/w two abelian categories. If $\mathcal{A}$ has enough projectives, we can construct the left derived functors L_iF $(i \geq 0)$ of F as follows

$$L_iF : \mathcal{A} \to \mathcal{B}$$

If $A \in \mathcal{A}$, choose a projective resolution $P_\cdot \to A$, where $P_\cdot = \cdots \to P_2 \to P_1 \xrightarrow{d_1} P_0 \to 0$

ie; $\cdots \to P_1 \to P_0 \xrightarrow{\varepsilon} A \to 0$ is exact

Now apply F to the chain complex $P_\cdot$,

we've $F(P_\cdot) = \cdots \to F(P_1) \xrightarrow{F(d_1)} F(P_0) \to 0$

Define $L_iF(A) = H_i(F(P))$

Q) Is $L_i F$ well defined ?

Q) Is $L_i F$ a functor ?

Note that

$$L_0 F\,(A) = H_0(F(P)) = \frac{F(P_0)}{Im(F(d_1))}$$

But we know that

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\varepsilon)} F(A) \to 0 \text{ is exact}$$

$\therefore\ Im(F(d_1)) \cong ker(F(\varepsilon))$

and $\dfrac{F(P_0)}{Im(F(d_1))} \cong \dfrac{F(P_0)}{ker(F(\varepsilon))} \cong Im(F(\varepsilon)) \cong F(A)$

Therefore $L_0 F\,(A) = F(A)$

<u>Lemma</u> The objects $L_i F(A)$ of $\mathcal{B}$ are well defined upto natural isomorphism

ie; If $Q \to A$ is a second projective

resolution, then there is a canonical isomorphism

$$L_i F(A) = H_i(F(P)) \xrightarrow{\cong} H_i(F(Q)).$$

<u>Proof</u>: By comparison theorem, there is a chain map $f: P \to Q$ lifting the identity map id_A,

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "f_2"] & P_1 \arrow[r] \arrow[d, dashed, "f_1"] & P_0 \arrow[r] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, "id_A"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

yielding a map $f_*: H_i F(P) \to H_i F(Q)$.

Note that any another such map $f': P \to Q$ is chain homotopic to f, so $f_* = f'_*$. [use the fact that F is additive]

Ily, there is a chain map $g: Q \to P$ lifting id_A and a map $g_*: H_i F(Q) \to H_i F(P)$.

Since $g \circ f$ and id_P are both chain maps from P to P lifting id_A, we've

$$g_* \circ f_* = (g \circ f)_* = (id_P)_* = id_{H_i F(P)}$$

Illy, $f_* \circ g_* = id_{H_i F(Q)}$

$\Rightarrow$ f_* & g_* are isomorphisms.

<u>Corollary</u>: If A is projective, then $L_i F(A) = 0$ for $i \neq 0$.

<u>Proof</u>: Take

$$\cdots 0 \longrightarrow 0 \longrightarrow A \xrightarrow{id_A} A \longrightarrow 0$$

<u>Lemma</u> If $f: A' \longrightarrow A$ is any map in $\mathcal{A}$, there is a natural map

$$L_i F(f): L_i F(A') \longrightarrow L_i F(A) \text{ for}$$

each i.

<u>Proof</u>: Let $P' \longrightarrow A'$ and $P \longrightarrow A$ be the

chosen projective resolutions. The comparison theorem yields a lift of f to a chain map $\tilde{f} : P' \to P$, hence a map $\tilde{f}_* : H_iF(P') \to H_iF(P)$.

Any other lift is chain homotopic to $\tilde{f}$, so $\tilde{f}_*$ is independent of the choice of $\tilde{f}$.

The map $L_iF(f) = \tilde{f}_*$.

<u>Thm</u>: $L_iF : \mathcal{A} \to \mathcal{B}$ is an additive functor.

<u>Proof</u>:

$$L_iF : \mathcal{A} \to \mathcal{B}$$

$$A \mapsto H_i(F(P))$$

$$\begin{pmatrix} A \\ f \downarrow \\ B \end{pmatrix} \longmapsto \begin{pmatrix} H_iF(P) \\ \tilde{f}_* \downarrow \\ H_iF(Q) \end{pmatrix}$$

If $A' \xrightarrow{f} A \xrightarrow{g} A''$ is given, then by defn $\widetilde{g \circ f} \simeq \tilde{g} \circ \tilde{f} \Rightarrow \widetilde{g \circ f}_* = \tilde{g}_* \circ \tilde{f}_*$.

Also $A \xrightarrow{id_A} A$ lifts to

$$id_{f(P)} : f(P) \to f(P)$$

$$\& \; id_{H_i f(P)} : H_i(f(P)) \to H_i(f(P))$$

$\therefore$ $L_i f$ is a functor.

For $f_1, f_2 : A' \to A$ in $\mathcal{A}$, we've

$f_1 + f_2 : A' \to A$ and note that

$\tilde{f}_1 + \tilde{f}_2$ and $\widetilde{f_1 + f_2}$ lifts $f_1 + f_2$,

therefore $\tilde{f}_1 + \tilde{f}_2 \simeq \widetilde{f_1 + f_2}$

$$\Rightarrow (\tilde{f}_1 + \tilde{f}_2)_* = (\widetilde{f_1 + f_2})_*$$

$$\tilde{f}_{1*} + \tilde{f}_{2*}$$

<u>Thm</u>: The derived functors $L_* F$ form a homological δ-functor.

Proof: For $f: A \to B$, we've

$$\left(L_i f : A \to B \right)_{i \geq 0}$$ additive functors.

Given any short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

and for projective resolutions $P' \to A'$ and $P'' \to A''$, by Horseshoe lemma, there is a projective resolution $P \to A$ fitting into a short exact sequence

$$0 \to P' \to P \to P'' \to 0$$

ie,

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, dashed] & A'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r, dashed] & P_0' \arrow[r, dashed] \arrow[u] & P_0 \arrow[r, dashed] \arrow[u, dashed] & P_0'' \arrow[r, dashed] \arrow[u] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & \vdots \arrow[d, dashed, leftarrow] & \vdots \arrow[d, dashed, leftarrow] & \vdots \arrow[d, dashed, leftarrow] & \\
0 \arrow[r, dashed] & P_1' \arrow[r, dashed] & P_1 \arrow[r, dashed] & P_1'' \arrow[r, dashed] & 0 \\
 & \vdots \arrow[u, dashed] & \vdots \arrow[u, dashed] & \vdots \arrow[u, dashed] &
\end{tikzcd}$$

we've

$$\begin{tikzcd}
0 \arrow[r] & P_n' \arrow[r] & P_n \arrow[r] & P_n'' \arrow[r] \arrow[l, dashed, bend left] & 0
\end{tikzcd}$$

split exact.

[since P_n'' is projective]

$$\left[\begin{tikzcd}
 & P_n'' \arrow[d, "id"] \arrow[dl, dashed] & \\
P_n \arrow[r] & P_n'' \arrow[r] & 0
\end{tikzcd}\right]$$

Since F is right exact, it is additive

[F preserves colimits

$\Rightarrow F(A \oplus B) \cong F(A) \oplus F(B)$

$\Rightarrow F$ additive]

and hence preserves split exact

sequences.

$$\therefore \quad 0 \to F(P_n') \to F(P_n) \to F(P_n'') \to 0$$

is split exact $\forall n$.

In particular

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is exact.

Writing out the corresponding long exact sequence, we get

$$\begin{tikzcd} \cdots \arrow[r] & H_i F(P') \arrow[r] & H_i F(P) \arrow[r] & H_i F(P'') \arrow[dll, bend left] \\ & H_{i-1} F(P') \arrow[r] \arrow[d, equal] & H_{i-1} F(P) \arrow[r] & \cdots \\ & L_{i-1} F(A') & & \end{tikzcd}$$

ie, associated to any short exact sequence,

$$0 \to A' \to A \to A'' \to 0,$$

we've

$$\delta_i : L_i f(A'') \longrightarrow L_{i-1}(f(A'))$$

such that

$$\cdots \to L_i f(A'') \xrightarrow{\delta_i} L_{i-1} f(A') \to L_{i-1} f(A) \to L_{i-1} f(A'') \xrightarrow{\delta_{i-1}} L_{i-2} f(A') \to \cdots$$

is exact.

We need to check naturality of δ_i

(use comparison theorem
(technical proof))

Thm: Assume that $\mathcal{A}$ has enough projectives. Then for any right exact functor $F : \mathcal{A} \to \mathcal{B}$, the derived functor $(L_i F)_{i \geq 0}$ form a universal δ-functor.

2.5 Right Derived functors

Defn: Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor b/w two abelian categories.

If $\mathcal{A}$ has enough injectives, we can construct right derived functors,

$$R^i F_{(i \geq 0)} \text{ of } F \text{ as follows:}$$

$$R^i F : \mathcal{A} \longrightarrow \mathcal{B}$$

Let $A \in \mathcal{A}$, take an injective resolution $A \longrightarrow I^{\bullet}$.

ie; we've

$$0 \to A \to I^0 \to I^1 \to \cdots \quad \text{exact.}$$

If we apply f to I°, we've the cochain complex

$$f(I^\cdot) : \quad 0 \to f(I^0) \to f(I^1) \to \cdots$$

Define $R^i f(A) = H^i(f(I))$

Note that

$$R^0 f(A) = H^0 f(I) = \ker f(d^0) \cong f(A)$$

$$\left[\; 0 \to f(I^0) \xrightarrow{f(d^0)} f(I^1) \to f(I^2) \cdots \right.$$

$$\left. 0 \to f(A) \to f(I^0) \to f(I^1) \text{ exact} \right]$$

<u>Rmk</u>: If $f : \mathcal{A} \to \mathcal{B}$ is left exact with $\mathcal{A}$ having enough injectives, then $f^{op} : \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$ is right exact with

$\mathcal{A}^{op}$ having enough projectives.

Therefore we can construct the left derived functors $L_i F^{op}$ as well; we see that

$$R^i F(A) = (L^i F^{op})^{op}(A).$$

$\therefore$ All results about left derived functors apply to right derived functors.

<u>Defn</u>: For each R-module A, the functor $F(B) = \mathrm{Hom}_R(A, B)$ is left exact. Its right derived functors are called Ext groups.

$$\mathrm{Ext}^i_R(A, B) = R^i \mathrm{Hom}_R(A, -)(B)$$

ie; take an injective resolution $I^{\cdot}$

of B,

ie; $0 \to B \to \underline{I}^0 \to I^1 \to \cdots$ is exact.

$$\mathrm{Ext}^i_R(A,B) = R^i \mathrm{Hom}_R(A,-)(B)$$
$$= H^i(\mathrm{Hom}_R(A,I)).$$

In particular

$$\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}_R(A,B).$$

<u>Defn</u>: Let A be an R-module. The functor $A \otimes \_ : R\text{-mod} \longrightarrow R\text{mod}$ is right exact.

Its left derived functors are called Tor groups.

$$\mathrm{Tor}^R_n(A,B) = L_n(A \otimes \_)(B)$$

<u>Fact</u>: $\mathrm{Ext}^n_R(A,B) = R^n \mathrm{Hom}_R(A,-)(B)$

$$\cong R^n \mathrm{Hom}_R(-, B)(A) \quad \forall n$$

$$\begin{array}{ccc} \mathrm{Tor}_n^R(A,B) & \cong & \mathrm{Tor}_n^R(B,A) \\ \| & & \| \\ L_n(A \otimes -)(B) & & L_n(B \otimes -)(A) \\ & & \| \\ & & L_n(- \otimes_R B)(A) \end{array}$$

3.1 Tor for Abelian groups

Let B be an abelian group.

Calculate $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}_p, B)$?

We've the projective resolution:

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

ie; we've the chain complex,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow 0 .$$

Apply $- \otimes B$ to it, we get

$$0 \to B \xrightarrow{p} B \to 0.$$

$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}_p, B) = B/pB$$

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, B) = {}_pB = \{ b \in B \mid pb = 0 \}$$

$$\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}_p, B) = 0 \quad \text{for } n \geq 2.$$

Prop: For all abelian groups A and B

a) $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group.

b) $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for $n \geq 2$

Proof: Suppose A is a finitely generated group.

Then $A \cong \mathbb{Z}^m \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_r}$ for $m, p_1, \ldots, p_r \in \mathbb{Z}$

Since $\mathbb{Z}^m$ is projective,

$\mathrm{Tor}_n(\mathbb{Z}^m, -)$ vanishes for $n \neq 0$.

Note that Tor_n is an additive functor.

$\therefore$ It preserves direct sums.

ie, $\mathrm{Tor}_n(A,B) \cong \mathrm{Tor}_n(\mathbb{Z}_{p_1}, B) \oplus \cdots$

$$\oplus \mathrm{Tor}_n(\mathbb{Z}_{p_r}, B).$$

$\therefore$ $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ is a torsion abelian group

and $\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = 0$ for $n \geq 2$.

Now, In general, any abelian group A is the direct limit of its finitely generated subgroup,

$$A = \varinjlim A_\alpha$$

$$\mathrm{Tor}_n(A,B) = \mathrm{Tor}_n(\varinjlim A_\alpha, B) \cong \varinjlim \mathrm{Tor}_n(A_\alpha, B).$$

Because direct limit of torsion abelian groups are torsion abelian, we get the result.

<u>Prop</u> $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion

subgroup of B for every abelian group B.

<u>Proof</u>: $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) \cong \varinjlim_{p \in \mathbb{Z}} \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, B)$

$$\cong \varinjlim ({}_pB) = \bigcup_p \{b \in B \mid pb = 0\}$$

which is a torsion subgroup of B.

<u>Prop</u>: If A is a torsion free abelian group, then $\operatorname{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for $n \neq 0$ and all abelian groups B.

<u>Proof</u>: If A is torsion free, then it is isomorphic to $\mathbb{Z}^m$ for some m.

$\therefore$ $\operatorname{Tor}_n(A, B) \cong \varinjlim \operatorname{Tor}_n(\mathbb{Z}^m, B) = 0$

<u>Corollary</u>

For every abelian group A,

$\mathrm{Tor}_1^{\mathbb{Z}}(A, -) = 0 \iff A$ is torsion free

$\iff \mathrm{Tor}_1^{\mathbb{Z}}(-, A) = 0$

<u>Proof</u>: $\Leftarrow$ ✓

$\Rightarrow$: $\mathrm{Tor}_1^{\mathbb{Z}}(A, -)(\mathbb{Z}_p) = 0 \quad \forall\, p \geq 2$

$\Rightarrow\ {}_pA = 0 \quad \forall\, p \geq 2$

$\Rightarrow$ A is torsion free.