Chapter 3

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§ 3.2 - Tor and Flatness

Notation: R-mod (left) (usually B) mod-R (right) (usually A)

Recall: $B \in R$-mod $\rightsquigarrow$ $- \otimes_R B$: mod-$R \to Ab$ is right exact

$$\cdots \to P_1 \to P_0 \to A \to 0 \quad \text{proj res.}$$

apply $-\otimes_R B$

$$\cdots \to P_1 \otimes_R B \to P_0 \otimes_R B \to \cancel{A \otimes_R B} \to 0 \quad \text{chain complex} \quad (\star)$$

$\rightsquigarrow$ $Tor_n^R(A,B)$ is the homology.

Def $B \in R$-mod is exact if $- \otimes_R B$ is exact.

$A \in$ mod-R is exact if $A \otimes_R -$ is exact.

Fact B flat $\iff$ $Tor_n^R(A,B) = 0$ $\forall n \neq 0$ $\forall A \in$ mod-R.

Proof:

($\Rightarrow$) Trivial, since $\star$ is exact...

($\Leftarrow$) For any exact sequence $0 \to A \to A' \to A'' \to 0$ we have a long exact seq. $\cdots \to Tor_1^R(A,B) \to A \otimes_R B \to A' \otimes_R B \to A'' \otimes_R B \to 0$

Thus, $Tor_1^R(A,B) = 0 \Rightarrow - \otimes_R B$ exact.

Rmk: projective $\Rightarrow$ flat.

Idea: B proj $\Rightarrow$ $\cdots \to 0 \to B \xrightarrow{id} B \to 0$ proj. resol. $\Rightarrow \mathrm{Tor}_n^R(A,B)=0$ $\forall n \neq 0\ \forall A \in$ mod-R.

But, Flat $\not\Rightarrow$ projective (e.g., $R=\mathbb{Z}$, $B=\mathbb{Q}$).

Idea: $\mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q})=0$ because $\mathbb{Q}$ is torsion free, and for $n \geq 2$ $\mathrm{Tor}_n^{\mathbb{Z}}(A,\mathbb{Q})=0$ $\forall A \in \underline{Ab}$.
But $\mathbb{Q}$ is not projective because it's not a direct summand of a free $\mathbb{Z}$-module!

Def (Potrjagin dual) $B \in R$-mod, $B^* = \mathrm{Hom}_{Ab}(B, \mathbb{Q}/\mathbb{Z})$ is a right R-mod via $(fr)(b) := f(rb)$.

Prop $B \in R$-mod. TFAE:
(i) B flat (ii) B^* injective (iii) $\mathrm{Tor}_1^R(R/I, B)=0$ $\forall I \subseteq R$ right ideal.

the proof uses the Potrjagin dual.

Thm Every finitely presented flat R-module is projective.

Prop (Flat Resolution) $\mathrm{Tor}_*(A,B)$ may be computed using a resolution of flat modules (e.g., $F_\bullet \to A$).

Rmk: There are some results about localization on § 3.2.

§ 3.3 - Ext for Nice Rings

Rmk: $A \in \mathbb{Z}\text{-mod}$ (Ab) is injective $\Leftrightarrow$ A is divisible
$[\forall n \in \mathbb{Z}^{\times}\ \forall a \in A\ \exists b \in A$ s.t. $a = nb]$

Lemma: $\mathrm{Ext}^n_{\mathbb{Z}}(A,B) = 0 \quad \forall n \geq 2 \ \forall A, B \in Ab$.

Proof: Consider $0 \to B \xhookrightarrow{\eta} I^0$, with I^0 injective.
Since $I^1 := I^0/\eta(B)$ is divisible (it is a quotient of a divisible),
we have a inj. resol. $0 \to B \to I^0 \to I^1 \to 0 \to \cdots$
Hence, $\mathrm{Ext}^*_{\mathbb{Z}}(A,B)$ is the cohomology of

$$0 \to \mathrm{Hom}(A, I^0) \to \mathrm{Hom}(A, I^1) \to 0 \to \cdots \quad \square$$

Example: $A = \mathbb{Z}/p\mathbb{Z}$

Consider $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ proj. resol.

and remember $\mathrm{Hom}(\mathbb{Z}, B) \cong B$.

apply $\mathrm{Hom}(-,B)$: $\quad 0 \to B \xrightarrow{p} B \to 0 \to \cdots$

Therefore, $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \cong {}_pB$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \cong B/pB$

Example $B = \mathbb{Z}$. $\rightsquigarrow$ $0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ inj. resol.

A torsion grp

apply $\mathrm{Hom}(A, -)$ $\rightsquigarrow$ $0 \to \mathrm{Hom}(A, \mathbb{Q}) \xrightarrow{\pi_*} \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to 0 \to \cdots$

Therefore, $\mathrm{Ext}^0_{\mathbb{Z}}(A, \mathbb{Z}) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = A^*$.

Prop (i) $\mathrm{Ext}^n_R(\bigoplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}^n_R(A_\alpha, B)$.

(ii) $\mathrm{Ext}^n_R(A, \prod_\beta B_\beta) \cong \prod_\beta \mathrm{Ext}^n_R(A, B_\beta)$.

Idea: The proof boils down to the facts $\mathrm{Hom}(\bigoplus_\alpha P_\alpha, B) \cong \prod_\alpha \mathrm{Hom}(P_\alpha, B)$

& $\mathrm{Hom}(A, \prod_\beta I_\beta) \cong \prod_\beta \mathrm{Hom}(A, I_\beta)$.

Example $\mathbb{Z}$ is projective $\Rightarrow$ $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, B) = 0$ $\forall n \neq 0$.

A f.g. abelian group, i.e., $A \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n\mathbb{Z}$

$$\Rightarrow \mathrm{Ext}^*_{\mathbb{Z}}(A, B) \cong \bigoplus_{i=1}^n \mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/p_i\mathbb{Z}, B).$$

Rmk: When R is commutative $\mathrm{Ext}^*_R(A, B)$ is a R-module.

Rmk: When R is noetherian, we have a localization theorem for Ext.

§ 3.4 - Ext and Extensions

<u>Def</u> We say that a exact seq. $\xi : 0 \to B \to X \to A \to 0$ is an <u>extension</u> of A by B.

Two ξ and ξ' are <u>equivalent</u> if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

An extension is <u>split</u> if it is equiv. to $0 \to B \to A \oplus B \to A \to 0$

<u>Lemma</u> $\mathrm{Ext}^1(A,B) = 0 \implies$ every extension of A by B is split.

<u>Proof</u>: $\xi : 0 \to B \to X \xrightarrow{\pi} A \to 0$ extension

apply $\mathrm{Hom}(A,-)$ $\rightsquigarrow$ $0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,X) \xrightarrow{\pi_*} \mathrm{Hom}(A,A) \xrightarrow{\partial} \underbrace{\mathrm{Ext}^1(A,B)}_{=0} \to \cdots$

$\implies \pi_*$ is surjective $\implies \exists\, \lambda \in \mathrm{Hom}(A,X)$ s.t. $\pi_*(\lambda) = \pi \circ \lambda = id_A$. ■

<u>Moral</u> (of the proof) The class of $\partial(id_A) \in \mathrm{Ext}^1(A,B)$ is an obstruction to ξ being split!

$$\begin{tikzcd} & A \arrow[dl, dashed, "\lambda"'] \arrow[d, "id_A"] & \\ X \arrow[r, "\pi"'] & A \arrow[r] & 0 \end{tikzcd}$$

Thm
$$\left\{\begin{array}{l}\text{equiv. classes of}\\ \text{ext. of } A \text{ by } B\end{array}\right\} \xleftrightarrow{1:1} \operatorname{Ext}^1(A,B).$$
$$[\xi] \overset{\Phi}{\longmapsto} \partial(\mathrm{id}_A).$$

Sketch Surjective: Take $x \in \operatorname{Ext}^1(A,B)$.

Fix $0 \to M \xrightarrow{i} P \to A \to 0$ s.e.s with P projective.

$$\xrightarrow{\operatorname{Hom}(-,B)} \operatorname{Hom}(P,B) \longrightarrow \operatorname{Hom}(M,B) \xrightarrow{\partial} \operatorname{Ext}^1(A,B) \longrightarrow 0 \;(= \operatorname{Ext}^1(P,B))$$
$$\beta \longmapsto x$$
$\beta \rightsquigarrow$ chosen

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\beta"'] \arrow[rd, phantom, "\text{pushout}"] & P \arrow[r] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi : 0 \arrow[r] & B \arrow[r, dashed, "\iota"'] & X \arrow[r, dashed] & A \arrow[r] & 0
\end{tikzcd}$$

Rmks: • X is the pushout.
• $X \to A$ defined using $B \xrightarrow{0} A$ and $P \to A$.

Naturality of $\partial \Rightarrow \Phi([\xi]) = x$.

also, Ψ is well-defined!

Injective: Define $\Psi(x) := [\xi]$, then we can show that $\Psi(\Phi([\xi])) = [\xi]$. $\square$

Def (Baer sum) $\xi : 0 \to B \xrightarrow{i} X \xrightarrow{\pi} A \to 0$ ext.
$\xi' : 0 \to B \xrightarrow{i'} X' \xrightarrow{\pi'} A \to 0$

$X'' := \{(x,x') \in X \times X' : \pi(x) = \pi'(x')\}$ is the pullback
$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
X' \arrow[r, "\pi'"'] & A
\end{tikzcd}$$

$\tilde{\Delta} B := \{(-i(b), i'(b)) : b \in B\} \rightsquigarrow Y := X''/\tilde{\Delta} B$

Then $\xi'' : 0 \to B \to Y \to A \to 0$ is the Baer sum.
$$b \mapsto (i(b), 0)$$
$$[(x,x')] \mapsto \pi(x)$$

Rmk: We can define $\operatorname{Ext}^n(A,B)$ using extensions
$$0 \to B \to X_n \to \cdots \to X_1 \to A \to 0. \quad \text{(Yoneda Construction)}$$

Example $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \dfrac{\mathbb{Z}/p\mathbb{Z}}{p(\mathbb{Z}/p\mathbb{Z})} \cong \mathbb{Z}/p\mathbb{Z} \rightsquigarrow$ we have p extensions!

- The split $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$
- For each $n = 1, \dots, p-1$: $0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{n} \mathbb{Z}/p\mathbb{Z} \to 0$

§3.6 - Universal Coefficient Theorems (UCT)

Thm (Künneth Formula) $P_\bullet$ chain complex of flat right R-mod, s.t. $d(P_n) \subseteq P_{n-1}$ is flat $\forall n$, $M \in R\text{-}\underline{\mathrm{mod}}$. $\Longrightarrow \exists$ s.e.s.

$$(\star)\quad 0 \longrightarrow H_n(P) \otimes_R M \longrightarrow H_n(P \otimes M) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(P), M) \longrightarrow 0 \quad (\forall n)$$

Proof: Since $\exists\ 0 \to Z_n \to P_n \to d(P_n) \to 0 \Longrightarrow Z_n$ is flat

Moreover, $\mathrm{Tor}_1^R(d(P_n), M) = 0 \Longrightarrow 0 \to Z_n \otimes M \to P_n \otimes M \to d(P_n) \otimes M \to 0$ exact.

$\Longrightarrow$ we have a chain complex of exact sequences.

Note that the differentials on $Z_\bullet \otimes M$ and $d(P_\bullet) \otimes M$ are zero.

$$\Longrightarrow H_{n+1}(dP \otimes M) \xrightarrow{\partial_{n+1}} H_n(Z \otimes M) \to H_n(P \otimes M) \to H_n(dP \otimes M) \xrightarrow{\partial_n} H_n(Z \otimes M) \to \cdots$$

$$H_{n+1}(dP \otimes M) \cong d(P_{n+1}) \otimes M,\quad H_n(Z \otimes M) \cong Z_n \otimes M,\quad H_n(dP \otimes M) \cong dP_n \otimes M,\quad H_n(Z \otimes M) \cong Z_n \otimes M$$

using the definition of ∂.

Note that $\partial = i \otimes \mathrm{id}_M$, where $i : d(P_{n+1}) \hookrightarrow Z_n$.

Also, $0 \to d(P_{n-1}) \longrightarrow Z_{n-1} \longrightarrow H_{n-1}(P) \to 0$ is a flat resolution.

$\Rightarrow \mathrm{Tor}_*(H_n(P), M)$ is the homology of $0 \to d(P_{n+1}) \otimes M \xrightarrow{\partial_n} Z_n \otimes M \to 0$,

i.e., $0 \longrightarrow \operatorname{coker} \partial_{n+1} \to H_n(P \otimes M) \to \ker \partial_n \to 0$

$\operatorname{coker} \partial_{n+1} \cong \mathrm{Tor}_0(H_n(P), M) \cong H_n(P) \otimes M$, $\quad \ker \partial_n \cong \mathrm{Tor}_1(H_{n-1}(P), M)$.

Thm (Universal Coefficient Thm for Homology) (UCT)

$P_\bullet$ chain complex of free ab. grps. Then (★) splits noncanonically and $H_n(P\otimes M) \cong (H_n(P)\otimes M) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M)$.

Proof: Recall, $A \leq B$, B free abelian $\Rightarrow$ A free abelian. → the seq. splits

Hence $d(P_n) \leq P_n$ is free and $P_n \cong Z_n \oplus d(P_n)$

~~$\xRightarrow{\otimes M}$ $P_n \otimes M = (Z_n\otimes M)\oplus(d(P_n)\otimes M)$.~~

$d_n \otimes id_M : P_n \otimes M \to P_{n-1}\otimes M$

Moreover, $Z_n \otimes M$ is a direct summand of $\mathrm{Ker}(d_n \otimes id_M)$.

$\Rightarrow$ $\dfrac{Z_n\otimes M}{(d_{n+1}\otimes id_M)(P_{n+1}\otimes M)}$ is a direct summand of $\dfrac{\mathrm{Ker}(d_n\otimes id_M)}{(d_{n+1}\otimes id_M)(P_{n+1}\otimes M)}$

$\cong H_n(P)\otimes M$ $\qquad\qquad$ $\cong H_n(P\otimes M)$

By the Künneth formula, the other summand is $\mathrm{Tor}_1^R(H_{n-1}(P), M)$.

Thm (UCT for Cohomology) $P_\bullet$ chain complex of proj. R-modules s.t. $d(P_n)$ is also proj. $\forall n$. Then, there exists a noncanonically split exact seq.

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P,M)) \longrightarrow \mathrm{Hom}_R(H_n(P), M) \longrightarrow 0$$