Homological Dimension

Motivation

- Give several definitions of dimensions of rings / modules
- Measure how far a ring is from being semisimple
- Applications to Algebraic Geometry

4.1 Dimensions

Def Let $A \in \text{Mod-}R$.

i) The projective dimension $pd(A)$ of A is the minimum n s.t.
$0 \to P_n \to \dots \to P_1 \to P_0 \to A \to 0$ is a projective resolution.

ii) The injective dimension $id(A)$ of A is the minimum n s.t.
$0 \to A \to E^0 \to E^1 \to \dots \to E^n \to 0$ is an injective resolution.

iii) The flat dimension $fd(A)$ of A is the minimum n s.t.
$0 \to F_n \to \dots \to F_1 \to F_0 \to A \to 0$ is a flat resolution.

(These are allowed to be infinite)

Ex

1) $\dots \to 0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ is a projective resolution.
↳ Can be shown this is of minimal length
$\Rightarrow pd(\mathbb{Z}/p\mathbb{Z}) = 1$.

2) Let P be a projective R-module.
$\dots \to 0 \to P \xrightarrow{id} P \to 0$ is a projective resolution $\Rightarrow pd(P) = 0$.

Rmk

Projective $\Rightarrow$ flat (every projective resolution is a flat resolution)
so $fd(A) \leq pd(A)$.

Ex

As a $\mathbb{Z}$-module, $fd(\mathbb{Q}) = 0$ but $pd(\mathbb{Q}) = 1$.
Flat resolution: $\dots \to 0 \to \mathbb{Q} \xrightarrow{id} \mathbb{Q} \to 0$.

pd lemma Let $A \in \text{Mod-}R$. TFAE

i) $pd(A) \le d$

ii) $Ext_R^n(A,B) = 0 \ \forall n > d, \forall B$

iii) $Ext_R^{d+1}(A,B) = 0 \ \forall B$

iv) If $0 \to M_d \to P_{d-1} \to P_{d-2} \to \dots \to P_1 \to P_0 \to A \to 0$ is any resolution with the P_i projective, then M_d is projective.

Dimension shifting

If $0 \to M_m \to P_m \to P_{m-1} \to \dots \to P_0 \to A \to 0$ is exact with the P_i projective then $L_{m+1}F(A) \cong Ker(F(M_m) \to F(P_m))$.

Fact A is a projective R module $\iff Ext_R^1(A,B) = 0 \ \forall B$.

Proof of pd lemma

(4) $\Rightarrow$ (1) obvious, (2) $\Rightarrow$ (3) obvious

(1) $\Rightarrow$ (2) Let $n > d$. Since $pd(A) \le d$, there exists a projective resolution $\dots \to 0 \to P_d \to \dots \to P_1 \to P_0 \to A \to 0$ of A. Calculate $Ext_R^n(A,B)$ via $0 \to Hom(P_0,B) \to \dots \to Hom(P_d,B) \to 0 \to \dots$, so $Ext_R^n(A,B) = 0/0 = 0$.

(3) $\Rightarrow$ (4) Let $0 \to M_d \to P_{d-1} \to P_{d-2} \to \dots \to P_1 \to P_0 \to A \to 0$ be any resolution with the P_i projective. By Dimension Shifting, $Ext^{d+1}(A,B) \cong Ext^1(M_d,B)$. Then since $Ext^{d+1}(A,B) = 0$ by assumption, by the above fact, M_d is projective. ∎

id lemma Let $B \in \text{Mod-}R$. TFAE

i) $id(B) \le d$

ii) $Ext_R^n(A,B) = 0 \ \forall n > d \ \forall A$

iii) $Ext_R^{d+1}(A,B) = 0 \ \forall A$

iv) if $0 \to B \to E^0 \to \dots \to E^{d-1} \to M^d \to 0$ is any resolution with the E^i injective, then M^d is injective.

fd lemma Let $A \in$ Mod-R. TFAE

i) $fd(A) \leq d$

ii) $\operatorname{Tor}_n^R(A,B) = 0 \ \forall n > d \ \forall B \in R\text{-Mod}$

iii) $\operatorname{Tor}_{d+1}^R(A,B) = 0 \ \forall B \in R\text{-Mod}$

iv) If $0 \to M_d \to F_{d-1} \to \dots \to F_0 \to A \to 0$ is a res with the F_i flat, then M_d is also flat.

Lem $B \in R$-Mod is injective $\Leftrightarrow \operatorname{Ext}^1(R/I, B) = 0 \ \forall$ left ideals $I \subseteq R$.

Global Dimension Theorem Let R be a ring. The following are equal:

i) $\sup\{id(B) \mid B \in \text{Mod-}R\}$

ii) $\sup\{pd(A) \mid A \in \text{Mod-}R\}$

iii) $\sup\{pd(R/I) \mid I$ is a right ideal of $R\}$

iv) $\sup\{d \mid \operatorname{Ext}_R^d(A,B) \neq 0$ for some $A, B \in \text{Mod-}R\}$.

Def This number is the (right) global dimension of R.

Proof

pd & id lemmas $\Rightarrow \sup\{pd(A) \mid A \in \text{Mod-}R\} = d \Leftrightarrow \operatorname{Ext}_R^{d+1}(A,B) = 0 \ \forall A,B \in \text{Mod-}R$ $\Leftrightarrow \sup\{id(B) \mid B \in \text{Mod}R\} = d \Leftrightarrow \sup\{d \mid \operatorname{Ext}_R^d(A,B) \neq 0$ for some $A, B \in \text{Mod-}R\} = d$

If $\sup\{pd(R/I) \mid \underset{\text{right ideal}}{I \subseteq R}\} = \infty$, we are done, so assume $\sup\{pd(R/I)\} = d < \infty$ so $\operatorname{Ext}_R^{d+1}(R/I, B) = 0 \ \forall B \in \text{Mod-}R$. Assume for contradiction that $id(B) > d$ for some $B \in \text{Mod-}R$. For this B, choose a resolution $0 \to B \to E^0 \to E^1 \to \dots \to E^{d-1} \to M \to 0$ with the E^i injective. But $\forall I$, $0 = \operatorname{Ext}_R^{d+1}(R/I, B) \cong \operatorname{Ext}_R^1(R/I, M)$, so by the lemma, M is injective so $id(B) \not> d$, so $\sup\{id(B) \mid B \in \text{Mod-}R\} \leq \sup\{pd(R/I) \mid I$ is a right ideal of $R\}$. But $R/I \in \text{Mod-}R$ so $\sup\{pd(A) \mid A \in \text{Mod-}R\} \geq \sup\{pd(R/I) \mid I \subseteq R\}$, so all these numbers are equal. $\blacksquare$

Ex $R = \Bbbk[x]/(x^2)$ has $\mathrm{gl\,dim}(R) = \infty$, e.g. $\exists$ the infinite resolution
$\cdots \xrightarrow{x} \Bbbk[x]/(x^2) \xrightarrow{x} \Bbbk[x]/(x^2) \longrightarrow \Bbbk \longrightarrow 0$

Tor-dimension Theorem Let R be a ring. The following numbers are equal.
i) $\sup\{fd(A) \mid A \in \text{Mod-}R\}$
ii) $\sup\{fd(R/J) \mid J$ is a right ideal of $R\}$
iii) $\sup\{fd(B) \mid B \in R\text{-Mod}\}$
iv) $\sup\{fd(R/I) \mid I$ is a left ideal of $R\}$
v) $\sup\{d \mid \mathrm{Tor}_d^R(A,B) \neq 0$ for some R-modules $A, B\}$

Def
This common number is called the Tor dimension (or weak dimension) of R.

Ex
Every field has global & Tor dimension zero.
$\mathrm{gl\,dim}(\mathbb{Z}) = 1 = \mathrm{Tor\,dim}(\mathbb{Z})$
For R = infinite product of fields, $\mathrm{Tor\,dim}(R) = 0$ but $\mathrm{gl\,dim}(R) > 0$.
Rmk
Since projective $\Rightarrow$ flat, $fd(A) \leq pd(A)$ $\forall A \in \text{Mod-}R$, so taking supremums $\Rightarrow$
$\mathrm{Tor\text{-}dim}(R) \leq r.\mathrm{gl\,dim}(R)$

Prop Let R be right Noetherian. Then
i) $fd(A) = pd(A)$ $\forall A \in \text{mod-}R$ (i.e. finitely generated R-modules)
ii) $\mathrm{Tor\text{-}dim}(R) = r.\mathrm{gl.\,dim}(R)$
iii) If R is also left Noetherian, then $r.\mathrm{gl\,dim}(R) = l.\mathrm{gl\,dim}(R)$.

Def R is (right) Noetherian if every (right) ideal is finitely generated, i.e. if every module R/I is finitely presented.

4.2 Rings of small dimension

Def

A ring R is (right) semisimple if every right ideal is a direct summand of R. ($\Leftrightarrow$ R is the direct sum of its minimal ideals).

Wedderburn–Artin Theorem

Let R be a semisimple ring. Then $R \cong \prod_{i=1}^{r} R_i$, $R_i = Mat_{n_i}(D_i)$, where the D_i are division rings.

Consequence

- Right semisimple is the same as left semisimple
- Every semisimple ring is left & right Noetherian.

Note simple $\not\Rightarrow$ semisimple, e.g. first Weyl algebra

Maschke's Theorem

Let G be a finite group st. $Char(\mathbb{K}) \nmid |G|$. Then $\mathbb{K}G$ is semisimple.

Fact

Every finitely presented flat R-module M is projective.

Thm, Let R be a ring. TFAE (for 'R-module' = left or right module)

i) R is semisimple

ii) R has (left and/or right) global dimension 0

iii) Every R-module is projective

iv) Every R-module is injective

v) R is Noetherian, and every R-module is flat

vi) R is Noetherian, and has Tor dimension 0.

Proof (Already shown (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv))

R semisimple $\Leftrightarrow$ every ses $0 \to I \to R \to R/I \to 0$ splits $\Leftrightarrow pd(R/I) = 0$ $\forall$ (right/left) ideals I so (ii) $\Leftrightarrow$ (i). (v) $\Leftrightarrow$ (vi) by definition. (i) $\Rightarrow$ (v), (iii) $\Rightarrow$ (v)

Assume (v). Let I be an ideal of R. Then R/I is finitely presented & flat by assumption. Therefore R/I is projective, so $R \to R/I$ splits, and I is a direct summand of R, i.e. R is semisimple. ☺

Def

A ring R is quasi-Frobenius if it is (left & right) Noetherian, and R is an injective (left & right) R-module.

Note This property is categorical, so is preserved by Morita equivalence

Ex $\mathbb{Z}/n\mathbb{Z}$ is quasi-Frobenius

Ex Every semisimple ring is quasi-Frobenius.

Thm Let R be a ring. TFAE.

i) R is quasi-Frobenius

ii) Every projective right R-module is injective

iii) Every injective right R-module is projective

iv) Every projective left R-module is injective

v) Every injective left R-module is projective.

Fact R quasi Frobenius $\Rightarrow$ R semisimple or $\operatorname{gldim} R = \infty$.

Def

A Frobenius algebra over a field $\mathbb{K}$ is a finite dimensional algebra R s.t. $R \cong \operatorname{Hom}_{\mathbb{K}}(R, \mathbb{K})$ as (right) R-modules.

Ex Frobenius algebras are quasi-Frobenius.

Ex All finite dimensional Hopf algebras are Frobenius algebras.

If $\mathbb{K}G$ is not semisimple (i.e. $\operatorname{char}(\mathbb{K}) \mid |G|$), what can we say about $\mathbb{K}G$?

Prop

If G is a finite group, $\mathbb{K}G$ is a Frobenius algebra.

(sketch)
Proof Let $R = \mathbb{k}G$.
$\dim_{\mathbb{k}}(\mathbb{k}G) = |G| < \infty$. wts $\mathbb{k}G \cong \mathrm{Hom}_{\mathbb{k}}(\mathbb{k}G, \mathbb{k})$ as right R-modules.
Define $f: R \to \mathbb{k}$ where $e \in G$ is the identity element.
$\sum_{g \in G} c_g g \mapsto c_e$
Define $\alpha: R \to \mathrm{Hom}_{\mathbb{k}}(R, \mathbb{k})$, where $\alpha(r): R \to \mathbb{k}$.
$r \mapsto \alpha(r)$ $\quad$ $x \mapsto f(rx)$

Check: α is an R-homomorphism.
Injective: Suppose $\alpha(r) = 0$ for $r = \sum_{g \in G} r_g g$. Then since each $r_g = f(rg^{-1}) = \alpha(r)(g^{-1}) = 0$, $r = 0$.
Surjective: $\dim_{\mathbb{k}} R = \dim_{\mathbb{k}}(\mathrm{Hom}_{\mathbb{k}}(R, \mathbb{k}))$

Def
A ring R is von Neumann regular if $\forall a \in R \ \exists x \in R$ s.t. $axa = a$.

Ex An infinite product of fields is von Neumann regular ($\Rightarrow$ not every von Neumann regular ring is semisimple).
Ex If V is a $\mathbb{k}$-vector space, $R = \mathrm{End}_{\mathbb{k}}(V)$ is Von Neumann regular.
R is semisimple $\Leftrightarrow \dim_{\mathbb{k}}(V) < \infty$.

Thm Let R be a ring. TFAE
i) R is von Neumann regular
ii) R has Tor dimension 0
iii) Every R-module is flat
iv) R/I is projective $\forall$ finitely generated ideals I.

Rmk
Since $\mathrm{Tordim}(R) \leq \mathrm{gl}\dim(R)$, Noetherian + vN regular $\Rightarrow$ semisimple.
Def A ring R is (right) hereditary if every right ideal is projective.
Rmk semisimple $\Rightarrow$ hereditary
Ex The path algebra of a quiver is left hereditary.
Thm
A ring R is right hereditary $\Leftrightarrow$ r. $\mathrm{gl}\dim(R) \leq 1$.
Proof
Use $0 \to I \to R \to R/I \to 0$

4.3 Change of Rings Theorems (dimensions of polynomial rings)

General Change of Rings Theorem

Let $f: R \to S$ be a ring homomorphism, and let A be an S-module. Then (as an R-module), $pd_R(A) \leq pd_S(A) + pd_R(S)$.

1st Change of Rings Thm

Let $x \in Z(R)$ (centre) be a nonzerodivisor. If $A \neq 0$ is a R/x-module with $pd_{R/x}(A) < \infty$, then $pd_R(A) = 1 + pd_{R/x}(A)$

2nd Change of Rings Thm

Let $x \in Z(R)$ be a nonzerodivisor. If A is an R-module and x is a nonzerodivisor on A (i.e. $a \neq 0 \Rightarrow xa \neq 0$), then $pd_R(A) \geq pd_{R/x}(A/xA)$

Cor $pd_{R[x]}(A[x]) = pd_R(A)$ $\forall$ R-modules A, for $A[x] := R[x] \otimes_R A$ an $R[x]$-module.

Thm

$gl\,dim(R[x_1, \dots, x_n]) = n + gl\,dim(R)$

Cor

If $\mathbb{k}$ is a field then $gl\,dim(\mathbb{k}[x_1, \dots, x_n]) = n$

4.4 Local Rings

Convention: in this section,
A ring R is local if it is a commutative Noetherian local ring R with a unique maximal ideal m. Denote the residue field of R by $k = R/m$.

Def

i) The Krull dimension of a ring R is the length d of the longest chain $p_0 \subset p_1 \subset \dots \subset p_d$ of prime ideals in R. Denote by $\dim(R)$. R local $\Rightarrow \dim(R) < \infty$

ii) The embedding dimension of a local ring R is $\operatorname{emb}\dim(R) = \dim_k(m/m^2) < \infty$.

Rmk $\dim(R) \leq \operatorname{emb}\dim(R)$ $\forall$ local rings R

Def

A local ring R is called a regular local ring if $\dim(R) = \operatorname{emb}\dim(R) = \dim_k(m/m^2)$

Ex The local coordinate rings of smooth algebraic varieties are regular local rings.

Ex $\mathbb{k}[x_1, \dots, x_n]_m$, $m = (x_1, \dots, x_n)$ is regular local of dimension n.

Def

i) If A is a fin gen R-module, a regular sequence on A (or A-sequence) is a sequence $(x_1, \dots, x_n)$, $x_i \in m$, s.t. $(a \neq 0 \Rightarrow x_1 a \neq 0)$, i.e. x_1 is a nonzerodivisor on A and each x_i $(i > 1)$ is a nonzerodivisor on $A/(x_1, \dots, x_{i-1})A$.

ii) The grade of A, $G(A)$ is the length of the longest regular sequence on A. (note $\forall$ local rings R, $G(R) \leq \dim(R)$). (Also called depth)

iii) R is Cohen-Macaulay if $G(R) = \dim(R)$.

Ex $\mathbb{Z}$, $\mathbb{k}[x_1, \dots, x_n]$, $\mathbb{k}[x]/(x^2)$ are Cohen Macaulay
not regular

Prop

A regular local ring is an integral domain.

Ex Let $R = \mathbb{k}[x_1, \dots, x_n]$, $P_{i-1} = \langle x_1, \dots, x_i \rangle$ which is a prime ideal, since $R/P_{i-1} \cong \mathbb{k}[x_{i+1}, \dots, x_n]$ is a domain. $P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_{n-1} \supsetneq 0$ is a prime chain of length n (with maximal length), so $\dim(\mathbb{k}[x_1, \dots, x_n]) = n$ ($\Rightarrow \mathbb{A}^n$ has dimension n as an affine scheme).

Fact $G(A) = 0 \implies \mathrm{Hom}_R(k, A) \neq 0$

Thm

If R is a local ring and $A \neq 0$ is a fin gen R-module, then every maximal A-sequence has length $G(A)$, and $G(A)$ is the smallest n s.t. $\mathrm{Ext}^n_R(k, A) \neq 0$.

Thm

If R is a local ring and A is a fin gen R-module, then
$\mathrm{id}(A) \leq d \iff \mathrm{Ext}^n_R(k, A) = 0 \ \forall n > d$.

Cor

$\mathrm{id}(A)$ is the largest n s.t. $\mathrm{Ext}^n_R(k, A) \neq 0$.

Prop

If R is a local ring with residue field k, then $\forall$ fin gen R-modules A and $\forall d \in \mathbb{Z}$, $\mathrm{pd}(A) \leq d \iff \mathrm{Tor}^R_{d+1}(A, k) = 0$, so $\mathrm{pd}(A)$ is the largest d s.t. $\mathrm{Tor}^R_d(A, k) \neq 0$.

Cor

If R is a local ring, then $\mathrm{gl\,dim}(R) = \mathrm{pd}_R(R/m)$

Proof $\mathrm{pd}(R/m) \leq \mathrm{gl\,dim}(R) = \sup_I \{\mathrm{pd}(R/I)\} \leq \mathrm{fd}(R/m) \leq \mathrm{pd}(R/m)$

Grade 0 Lemma

If R is local and $G(R) = 0$ (i.e. every element of m is a zero divisor on R) then $\forall$ fin gen R-modules A, either $\mathrm{pd}(A) = 0$ or $\mathrm{pd}(A) = \infty$.

Thm

A local ring R is regular $\iff \mathrm{gl\,dim}(R) < \infty$.
In this case, $G(R) = \dim(R) = \mathrm{emb.\,dim}(R) = \mathrm{gl\,dim}(R) = \mathrm{pd}_R(k)$

Cor

A regular ring is Cohen Macaulay

Thm

Every regular local ring is a UFD.

Motivation For an algebraic variety X, a point $x \in X$ is smooth $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is regular.

Cor

Let R be a regular local ring and $p \subseteq R$ any prime ideal. Then the localisation R_p is also a regular local ring.

Rmk

The non-homological proof of this is long and difficult.

Proof

$R_p = S^{-1}R$ for $S = R - p$ a multiplicative set.
Considering an $S^{-1}R$ module A as an R-module, $\exists$ a projective resolution $P_\cdot \to A$ of length $\leq \mathrm{gl\,dim}(R)$.
Since $S^{-1}R$ is a flat R-module and $S^{-1}A = A$, $S^{-1}P \to A$ is a projective $S^{-1}R$-module resolution of length $\leq \mathrm{gl\,dim}(R)$, which is finite since R is regular. Therefore $\mathrm{gl\,dim}(R_p) < \infty$, so R_p is regular.