Derived Categories Part 1

$$\begin{tikzcd}
Ch(\mathcal{A}) \arrow[r, two heads] \arrow[dr, "\text{harder}"'] & K(\mathcal{A}) \arrow[d, "\text{localisation}"] \\
& D(\mathcal{A})
\end{tikzcd}$$

1. $K(\mathcal{A})$

<u>Def</u>

Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$ is the quotient category of $Ch(\mathcal{A})$ with
cochain complexes
- objects: cochain complexes (objects of $Ch(\mathcal{A})$)
- morphisms: chain homotopy equivalence classes of maps in $Ch(\mathcal{A})$.

<u>Fact</u>
- $K(\mathcal{A})$ is well defined as a category
- $K(\mathcal{A})$ is an additive category in such a way that the quotient $Ch(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$ is an additive functor.

Write $K^b(\mathcal{A})$, $K^-(\mathcal{A})$, $K^+(\mathcal{A})$ corresponding to $Ch^b(\mathcal{A})$, $Ch^-(\mathcal{A})$, $Ch^+(\mathcal{A})$

<u>Note</u> Could also have considered chain complexes rather than cochain complexes when constructing K.

We know cohomology is a functor $Ch(\mathcal{A}) \to \mathcal{A}$.

<u>Lem</u>

The cohomology $H^*(C)$ of a cochain complex C induces a family of well-defined functors $H^i : K(\mathcal{A}) \to \mathcal{A}$.

<u>Proof</u> (sketch)

Show that $u^* : H^i(A) \to H^i(B)$ induced by $u : A \to B$ is independent of the chain homotopy equivalence class of u.

Universal Property

Let $F: Ch(\mathcal{A}) \to \mathcal{D}$ be any functor that sends chain homotopy equivalences to isomorphisms. Then F factors uniquely through $K(\mathcal{A})$.

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathcal{D} \\ K(\mathcal{A}) \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Def

Let $u: A \to B$ be a morphism in $Ch(\mathcal{A})$

Recall $\exists$ a ses $0 \to B \xrightarrow{v} \text{cone}(u) \xrightarrow{\delta} A[-1] \to 0$ in $Ch(\mathcal{A})$

(in degree n: $0 \to B^n \to A^{n+1} \oplus B^n \to A^{n+1} \to 0$)

- The strict triangle on u is the triple (u, v, δ) of maps in $K(\mathcal{A})$:

$$\begin{tikzcd} & \text{cone}(u) \arrow[dl, "\delta"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

- Fix cochain complexes A, B, C, with $u: A \to B$, $v: B \to C$, $w: C \to A[-1]$ in $K(\mathcal{A})$. Then (u, v, w) is an exact triangle on (A, B, C) if there is a commutative diagram in $K(\mathcal{A})$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f", "\cong"'] & B \arrow[r, "v"] \arrow[d, "g", "\cong"'] & C \arrow[r, "w"] \arrow[d, "h", "\cong"'] & A[-1] \arrow[d, "f[-1]", "\cong"'] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & \text{cone}(u') \arrow[r, "\delta"] & A'[-1] \end{tikzcd},$$

where
$$\begin{tikzcd} & \text{cone}(u') \arrow[dl, "\delta"'] & \\ A' \arrow[rr, "u'"] & & B' \arrow[ul, "v'"'] \end{tikzcd}$$
is a strict triangle on $u': A' \to B'$,

and $f, g, h \in K(\mathcal{A})$ are isomorphisms.

↳ idea: (u, v, w) is "isomorphic" to a strict triangle.

→ Can think of (u, v, w) as a triangle in $K(\mathcal{A})$:
$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

Fact

Given an exact triangle (u, v, w) on (A, B, C), the cohomology sequence $\cdots \xrightarrow{w^*} H^i(A) \xrightarrow{u^*} H^i(B) \xrightarrow{v^*} H^i(C) \xrightarrow{w^*} H^{i+1}(A) \xrightarrow{u^*} \cdots$ is exact (under the identification $H^i(A[-1]) \cong H^{i+1}(A)$).

Ex.

Since $\mathrm{cone}(0) = A \oplus A[-1]$ and $\mathrm{cone}(1) \cong 0$ in $K(\mathcal{A})$,

(*) $0 : A \to A$ gives the exact triangle
$$\begin{tikzcd} & A \oplus A[-1] \arrow[dl] & \\ A \arrow[rr, "0"] & & A \arrow[ul] \end{tikzcd}$$
and $A \xrightarrow{1} A$ gives
$$\begin{tikzcd} & 0 \arrow[dl] & \\ A \arrow[rr, "1"] & & A \arrow[ul] \end{tikzcd}.$$

(*) Ex if (u, v, w) is an exact triangle, then so are its "rotates".

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd} \qquad \begin{tikzcd} & A[-1] \arrow[dl, "-u[-1]"'] & \\ B \arrow[rr, "v"] & & C \arrow[ul, "w"'] \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "v"'] & \\ C[+1] \arrow[rr, "-w[1]"] & & A \arrow[ul, "u"'] \end{tikzcd}$$

$0 : A \to A$

$$0 \xrightarrow{\xi} A \xrightarrow{v} \mathrm{cone}(0) = A \oplus A[-1] \xrightarrow{\delta} A[-1] \to 0$$

$\wr$

$$\begin{tikzcd} & A \oplus A[-1] \arrow[dl, "\delta"'] & \\ A \arrow[rr, "0"] & & A \arrow[ul, "v"'] \end{tikzcd}$$

$$1 : A \to A \rightsquigarrow 0 \to A \to \underset{\substack{\cong \\ 0}}{\mathrm{cone}(1)} \to A[-1] \to 0$$

$\wr$

$$\begin{tikzcd} & 0 \arrow[dl] & \\ A \arrow[rr, "1"] & & A \arrow[ul] \end{tikzcd}$$

2 Triangulated categories

We want to generalise this notion of exact triangle.
Think of exact triangles as replacements for ses.

Def

- Let K be an additive category and $T: K \to K$ be an additive functor which is an automorphism of K. We call T the <u>translation functor</u> on K.

  Notation: $T^n(X) = X[n]$ for $X \in K$, $n \in \mathbb{Z}$.

- A triangle in K is a diagram $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$, written as

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd} \quad , \quad \text{or} \quad \begin{tikzcd} & C \arrow[dl, "{[1]}"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd}$$

- A morphism of triangles is a triple (f, g, h) giving a commutative diagram in K

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA' \end{tikzcd}$$

- An isomorphism of triangles is a morphism (f,g,h) s.t. f,g,h are isos.

Def

An additive category K is a <u>triangulated category</u> if it is equipped with an automorphism $T: K \to K$ and a distinguished family of triangles (u,v,w) (called <u>distinguished or exact triangles</u>) s.t.

<u>TR1a</u>: Any triangle isomorphic to a distinguished triangle is a distinguished triangle.

<u>TR1b</u>: $\forall X \in K$, $\begin{tikzcd} & 0 \arrow[dl] & \\ X \arrow[rr, "1"] & & X \arrow[ul] \end{tikzcd}$ is a distinguished triangle.

<u>TR1c</u>: $\forall f: X \to Y$ in K, $\exists$ a distinguished triangle $\begin{tikzcd} & Z \arrow[dl] & \\ X \arrow[rr, "f"] & & Y \arrow[ul] \end{tikzcd}$.

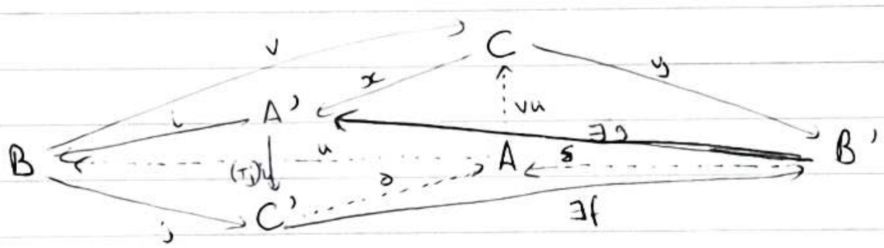
"every morphism can be embedded in a distinguished triangle".

<u>TR2</u>: (Rotation) $\begin{tikzcd} & Z \arrow[dl, "h"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$ is distinguished $\iff$ $\begin{tikzcd} & T(X) \arrow[dl, "-T(f)"'] & \\ Y \arrow[rr, "g"'] & & Z \arrow[ul, "h"'] \end{tikzcd}$ is distinguished.

(This is equivalent to the statement that both "rotates" of a distinguished triangle are distinguished).

TR3 : (Morphisms) Let
$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"] & T(X) \arrow[d, "T(u)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$
be a diagram where the rows are distinguished triangles and the first square commutes. Then $\exists$ $w : Z \to Z'$ s.t. the diagram including w is a morphism of distinguished triangles.

TR4 : (Octahedral axiom) Suppose there are three distinguished triangles
$$\begin{tikzcd}[column sep=small] & C' \arrow[dl, "\partial"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "j"'] \end{tikzcd} , \quad \begin{tikzcd}[column sep=small] & A' \arrow[dl, "i"'] & \\ B \arrow[rr, "v"'] & & C \arrow[ul, "x"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & B' \arrow[dl, "\delta"'] & \\ A \arrow[rr, "vu"'] & & C \arrow[ul, "y"'] \end{tikzcd} .$$
Then $\exists$ a distinguished triangle $\begin{tikzcd}[column sep=small] & A' \arrow[dl, "(Tj)i"'] & \\ C' \arrow[rr, "f"'] & & B' \arrow[ul, "g"'] \end{tikzcd}$ s.t. we get an octahedron where the 4 distinguished triangles form 4 of the faces and the remaining 4 faces commute, i.e. $\partial = \delta f$, $x = gy$, Also $yv = fj : B \to B'$ and $u\delta = ig : B' \to B$.



<u>Alternatively</u>: Let f, g, $h = g \circ f$ be morphisms in K. Then the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] & Y \arrow[r, "a"] \arrow[d, "g"'] & Z' \arrow[r] & T(X) \arrow[d, "T(id_X)"] \\ X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[r, "b"] \arrow[d, "id_Z"'] & Y' \arrow[r] & T(X) \arrow[d, "T(f)"] \\ Y \arrow[r, "g"] \arrow[d, "a"'] & Z \arrow[r, "c"] \arrow[d, "b"'] & X' \arrow[r] \arrow[d, "id_{X'}"'] & T(Y) \arrow[d, "T(a)"] \\ Z' \arrow[r, "u"'] & Y' \arrow[r, "v"'] & X' \arrow[r, "w"'] & T(Z') \end{tikzcd}$$
where the rows are distinguished triangles can be completed to the coloured diagram, where all 4 rows are distinguished triangles and the vertical arrows are morphisms of triangles.

Lem

If (u,v,w) is an exact triangle, then the compositions $v\circ u$, $w\circ v$ and $(Tu)\circ w$ are zero in K.

Proof (partial) For, $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{T(u)} TB$

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\ \downarrow 1 & & \downarrow u & & \downarrow 0 & & \downarrow 0 \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \end{array}$$

is a morphism of triangles, so $v\circ u = 1\circ v\circ u = 0\circ 0\circ 1 = 0$

Prop

$K(\mathcal{A})$ is a triangulated category.

Sketch of proof

Translation is $TA = A[-1]$. We have seen TR2, and part of TR1.

For TR3, suppose $C = \text{cone}(u)$, $C' = \text{cone}(u')$ (i.e. the triangles are strict).

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & \text{cone}(u) & \xrightarrow{w} & TA \\ f\downarrow & \circlearrowleft & \downarrow g & & \downarrow \exists h & & \downarrow Tf \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & \text{cone}(u') & \xrightarrow{w'} & TA' \end{array}$$

Then h exists by the naturality of the mapping cone construction.

For TR4, suppose the given triangles are strict, i.e. $C' = \text{cone}(u)$, $A' = \text{cone}(v)$, $B' = \text{cone}(v u)$. Define

$f^n : (C')^n = B^n \oplus A^{n+1} \longrightarrow (B')^n = C^n \oplus A^{n+1}$, $g^n : (B')^n = C^n \oplus A^{n+1} \to (A')^n = C^n \oplus B^{n+1}$

$(b,a) \longmapsto (v(b),a)$ $\qquad (c,a) \mapsto (c, u(a))$.

It can be shown that $\partial = \delta f$ and $x = gy$. Since the degree n part of $\text{cone}(f)$ is $(C^n \oplus A^{n+1}) \oplus (B^{n+1} \oplus A^{n+2})$, $\exists$ a natural inclusion γ of A' into $\text{cone}(f)$ giving

$$\begin{array}{ccccccc} C' & \xrightarrow{f} & B' & \xrightarrow{g} & A' & \xrightarrow{(Tj)i} & C'[-1] \\ \| & & \| & & \downarrow \gamma & & \| \\ C' & \to & B' & \to & \text{cone}(f) & \to & C'[-1] \end{array}$$

which commutes up to chain homotopy.

Check that γ is a chain homotopy equivalence.

It follows that $(f, g, (Tj)i)$ is an exact triangle, because it is isomorphic to the strict triangle of f. $\square$

Ex $K^b(\mathcal{A})$, $K^-(\mathcal{A})$, $K^+(\mathcal{A})$ are triangulated categories.

Def

A morphism of triangulated categories is an additive functor $F : K' \to K$ that commutes with the translation functor T and sends distinguished triangles to distinguished triangles

Def

Let K be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : K \to \mathcal{A}$ is called a (covariant) cohomological functor if whenever (u, v, w) is an exact triangle on (A, B, C), the les

$$\cdots \xrightarrow{w^*} H(T^i A) \xrightarrow{u^*} H(T^i B) \xrightarrow{v^*} H(T^i C) \xrightarrow{w^*} H(T^{i+1} A) \xrightarrow{u^*} \cdots$$

is exact in $\mathcal{A}$.

Notation: write $H^i(A) := H(T^i A)$, $H^0(A) := H(A)$

Ex

For $X \in K$, $\mathrm{Hom}_K(X, -) : K \to \mathrm{Ab}$ is a cohomological functor.

3 Localisation

Idea: Add in extra morphisms so that some of the morphisms in the original category become isomorphisms.

Compare to localising a ring: "adding in extra denominators/inverses".

Def

Let S be a collection of morphisms in a category $\mathcal{C}$. A <u>localisation of $\mathcal{C}$ wrt S</u> is a category $S^{-1}\mathcal{C}$, together with a functor $q: \mathcal{C} \to S^{-1}\mathcal{C}$ s.t.

i) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ $\forall s \in S$

ii) Any functor $F: \mathcal{C} \to \mathcal{D}$ s.t. $F(s)$ is an isomorphism $\forall s \in S$ factors uniquely through q.

Fact It follows that $S^{-1}\mathcal{C}$ is unique up to equivalence.

Ex

- Let S be the collection of chain homotopy equivalences in $Ch(\mathcal{A})$. Then by the universal property of $K(\mathcal{A})$, $K(\mathcal{A})$ is $S^{-1}Ch(\mathcal{A})$
- ~~Let $\hat{Q}$ be the collection of all quasi-isomorphisms in $Ch(\mathcal{A})$. Since $\hat{Q} \supseteq S$ from above,~~
- Let Q be the collection of quasi-isomorphisms in $K(\mathcal{A})$. Then $D(\mathcal{A}) := Q^{-1}K(\mathcal{A})$ is the derived category of $\mathcal{A}$. (**Def**)

Rmk When S is not a set, this causes some size problems, which we are ignoring.

$S^{-1}\mathcal{C}$ has objects $Ob(\mathcal{C})$.

For $M, N \in \mathcal{C}$, a path of length n is

$$M \xrightarrow{f_0} L_1 \xrightarrow{f_1} \cdots \xleftarrow{s_{i-1}} L_i \xrightarrow{f_i} \cdots \xleftarrow{s_{n-2}} L_{n-1} \xleftarrow{s_{n-1}} N \quad \text{for } s_j \in S$$

A morphism $M \to N$ in $S^{-1}\mathcal{C}$ is an equivalence class of paths between M & N under the equivalence $L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1} \sim L_{i-1} \xrightarrow{f_i \circ f_{i-1}} L_{i+1}$,

$$L \xrightarrow{s} P \xleftarrow{s} L \sim L \xrightarrow{id_L} L, \quad L \xleftarrow{s} P \xrightarrow{s} L \sim L \xrightarrow{id_L} L, \quad L \xrightarrow{id_L} L \xleftarrow{s} P \sim L \xleftarrow{s} P$$

Composition: concatenation

From now on, assume we can construct $S^{-1}\mathcal{C}$ (i.e. we don't have size problems).

Def

A collection S of morphisms in a category $\mathcal{C}$ is called a <u>multiplicative system</u> in $\mathcal{C}$ if it satisfies

i) S is closed under composition & contains all identity morphisms.

ii) (Ore condition) If $t: Z \to Y$ is in S, then $\forall\, g: X \to Y$ in $\mathcal{C}$, $\exists$ a diagram
$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "G"] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$
in $\mathcal{C}$ with $s \in S$.

Slogan "$t^{-1}g = fs^{-1}$". The symmetric statement also holds: "$fs^{-1} = t^{-1}g$".

iii) If $f, g: X \to Y$ in $\mathcal{C}$, then TFAE

a) $sf = sg$ for some $s \in S$ with source Y

b) $ft = gt$ for some $t \in S$ with target X.

Ex

Let R be an associative ring R with unit. Consider R as an additive category $\mathcal{R}$ with one object $*$. Let S be a subset of R which is closed under multiplication & contains 1. If R is commutative (or $S \subseteq Z(R)$), then S is a multiplicative system in $\mathcal{R}$ and $S^{-1}\mathcal{R} = S^{-1}R$

Fact

- If $\mathcal{C}$ is an additive category then so is $S^{-1}\mathcal{C}$, and q is an additive functor.
- If $\mathcal{A}$ is an abelian category, then so is $S^{-1}\mathcal{A}$, and q is exact.

$K^b(\mathcal{A}) \rightsquigarrow D^b(\mathcal{A})$, $K^+(\mathcal{A}) \rightsquigarrow D^+(\mathcal{A})$, $K^-(\mathcal{A}) \rightsquigarrow D^-(\mathcal{A})$

objects: cochain complexes which are bounded, bounded below, bounded above.

equivalent to previous definition of $S^{-1}\mathcal{A}$.

<u>Useful definition</u> Let $M, N \in \mathcal{A}$, S a multiplicative system. A (left) roof between M & N is a diagram
$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$
, where $s \in S$. $\sim$ denotes that this arrow is in S.

A right roof is
$$\begin{tikzcd} & L & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"', "\sim"] \end{tikzcd}$$
where $t \in S$.

Two roofs are equiv if $\exists H \in \mathcal{A}$, $p: H \to L$, $q: H \to K$ s.t.
$$\begin{tikzcd} & L \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ M & H \arrow[u, "p"] \arrow[d, "q"] & N \\ & K \arrow[ul, "\sim"] \arrow[ur, "g"'] & \end{tikzcd}$$
and $s \circ p = t \circ q \in S$.

4 Derived categories

Def

Let K be a triangulated category. The system S arising from a cohomological functor $H: K \to A$ is the collection of all morphisms s in K s.t. $H^i(s)$ is an isomorphism $\forall i$.

Universal property

Let $F: K \to L$ be a morphism of triangulated categories s.t. $F(s)$ is an isomorphism $\forall s \in S$, where S arises from a cohomological functor. Since $q: K \to S^{-1}K$ is a localisation, $\exists ! \, F': S^{-1}K \to L$ s.t. $F = F' \circ q$. In fact, F' is a morphism of triangulated categories.

Cor

$D(A)$, $D^b(A)$, $D^+(A)$, $D^-(A)$ are triangulated categories.

Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a ses in an abelian category A. Then we get a distinguished triangle in the derived category.

$$\begin{tikzcd} & D(N) \arrow[dl, "{[1]}"'] & \\ D(L) \arrow[rr, "D(f)"'] & & D(M) \arrow[ul, "D(g)"'] \end{tikzcd}$$

Ex

Suppose R is a hereditary ring. Then each $P \in D^b(\text{Mod-}R)$ is isomorphic to a finite sum of objects $S^n M$, $M \in \text{Mod-}R$, $n \in \mathbb{Z}$,

$(SM)_n = M_{n-1}$, and $\mathrm{Hom}_D(S^i M, S^j N) = \begin{cases} 0 & j \neq i, i+1 \\ \mathrm{Hom}_R(M,N) & j = i \\ \mathrm{Ext}^1_R(M,N) & j = i+1 \end{cases}$