So our construction this far:

localizing at quasi isomorphisms (Example 10.3.2)

$$\mathrm{Ch}(\mathcal{A}) \rightsquigarrow K(\mathcal{A}) \rightsquigarrow D(\mathcal{A})$$

localizing at chain homotopy equivalences — localizing at quasi-isomorphisms

↳ we really need this because our construction in TR 2 only commuted up to chain homotopy equivalence for example

What we gain by factoring through $K(\mathcal{A})$:

<u>the calculus of fractions:</u>

We want to invert morphisms in some multiplicative set S.
We set $\mathrm{Hom}_S(X,Y)$ to be the set of equivalence classes of left roofs $X \xleftarrow{s\in S} X_1 \xrightarrow{f} Y$. For this to be a set we need some extra considerations (see 10.3.6. from Weibel).

Now to get $S^{-1}\mathcal{C}$, we must provide a composition:

$$\mathrm{Hom}_S(Y,Z) \times \mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hom}_S(X,Z).$$

$$(Y \xleftarrow{t} Y_1 \xrightarrow{g} Z) \circ (X \xleftarrow{s} X_1 \xrightarrow{f} Y) \longmapsto (X \xleftarrow{st'} X_2 \xrightarrow{gf'} Z)$$

where

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "t'"'] & Y_1 \arrow[r, "g"] \arrow[d, "t\in S"] & Z \\ X & X_1 \arrow[l, "s"'] \arrow[r, "f"'] & Y \end{tikzcd}$$

st' is in S because S is closed under composition

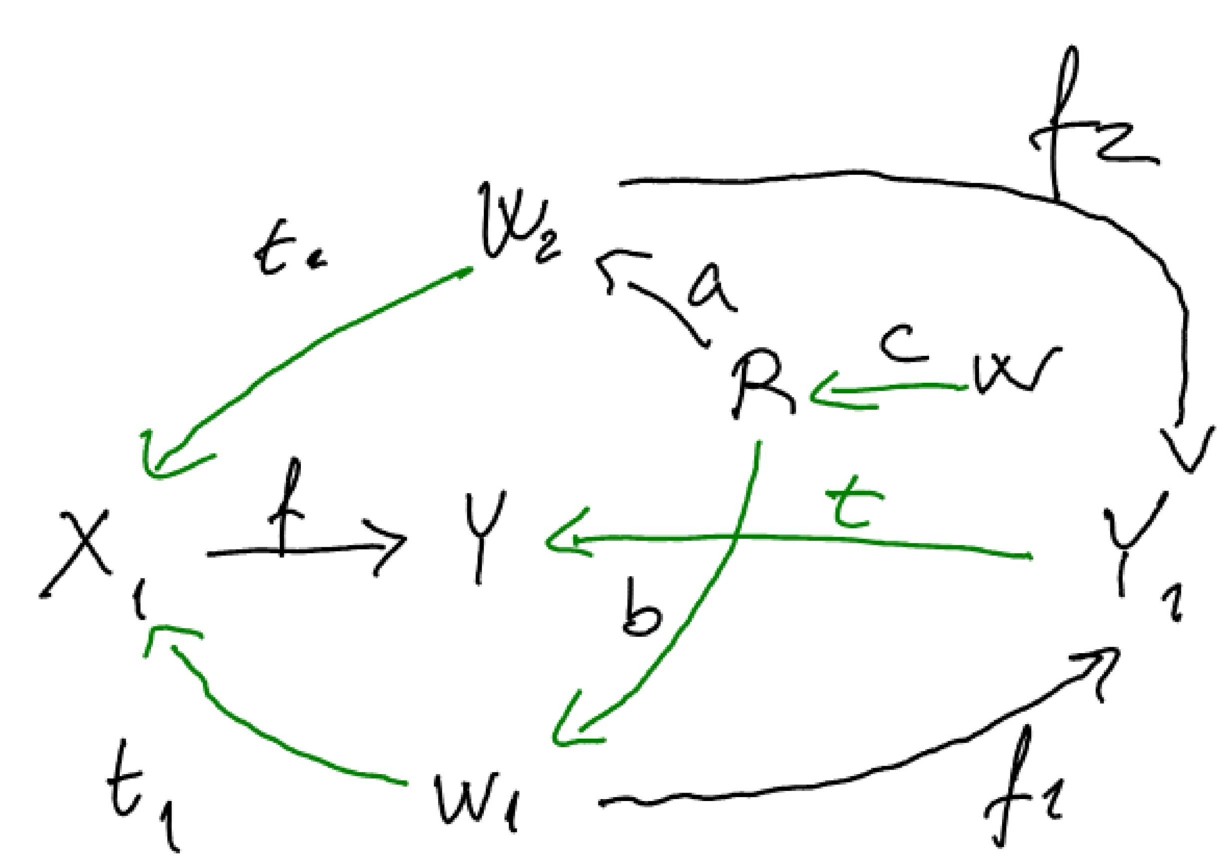
by the Ore condition such maps exist

independent of the choice of t' and f':

$$\begin{tikzcd} W_2 \arrow[rr, bend left, "f_2"] \arrow[dr, bend right, "t_2"'] & W_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & Y_1 \arrow[r, "g"] \arrow[d, "t"] & Z \\ X & X_1 \arrow[l, "s"'] \arrow[r, "f"'] & Y & \end{tikzcd}$$

we apply Ore to

$$\begin{tikzcd} R \arrow[r, "b"] \arrow[d, "a"'] & W_1 \arrow[d, "t_1"] \\ W_2 \arrow[r, "t_2"', "\cap S"] & X_1 \end{tikzcd}$$



so $f t_1 b = f t_2 a$ but we may not have $f_2 a = f_1 b$

but by (iii) $W \xrightarrow{c} R$ in S exists s.t. $f_2 ac = f_1 bc$ because $t f_2 a = t f_1 b$ and $t \in S$

Gabriel - Zisman Theorem: Let S be a locally small ($\forall X \in \mathcal{C}$ $\exists S_X$ set of morphisms in $\mathcal{C}$ s.t. $\forall X_1 \to X$ in S $\exists X_2 \to X_1$ in $\mathcal{C}$ with $X_2 \to X_1 \to X$ in S_X) multiplicative system of morphisms in $\mathcal{C}$. The category $S^{-1}\mathcal{C}$ with objects those of $\mathcal{C}$ and morphisms as in $\mathrm{Hom}_S(X,Y)$ exists and is the localization of $\mathcal{C}$ with respect to S. The localization functor sends a map $f: X \to Y$ to the fraction $X = X \xrightarrow{f} Y$ (and is identity on objects).

Corollary: Two maps $f, g: X \to Y$ become identified in $S^{-1}\mathcal{C}$ if and only if there is some $s \in S$ s.t. $sf = sg$.

Theorem: If $\mathcal{C}$ is an additive category then so is $S^{-1}\mathcal{C}$ and the localization functor is additive.

Def.: Let $\mathcal{B} \subseteq \mathcal{C}$ be a full subcategory and S a locally small multiplicative system in $\mathcal{C}$ such that $S \cap \mathcal{B}$ is a multiplicative system in $\mathcal{B}$. $\mathcal{B}$ is called a localizing subcategory if $S^{-1}\mathcal{B}\ (:= (S \cap \mathcal{B})^{-1}\mathcal{B}) \longrightarrow S^{-1}\mathcal{C}$ is fully faithful (induced by the inclusion).

If $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$ and $\forall c \in \mathcal{C}$ $\exists b \in \mathcal{B}$ and some $b \to c$ in S, then $S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}$.

Derived categories

Let K be a triangulated category, $H : K \to A$ a cohomological functor. The collection $S = \{ s \in \mathrm{Mor}(K) \mid H^i(s) \text{ is an isomorphism } \forall i \}$ we say is the system of morphisms arising from H.

<u>Proposition (10.4.1.)</u>: If S arises from a cohomological functor, then

① S is a multiplicative system.

② $S^{-1}K$ is a triangulated category and $K \to S^{-1}K$ is a morphism of triangulated categories.

<u>Proof:</u> ① (i) S contains the identities and is closed under composition ✓

(ii) Given $s : Z \to Y$ and $f : X \to Y$ such that $s \in S$, we can embed s into an exact triangle (by TR1c)

$$\begin{tikzcd}
W \arrow[r, "t"] \arrow[d, "g"] & X \arrow[r, "uf"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r, "v"] \arrow[d, equal] & W[-1] \arrow[d, "Tg"] \\
Z \arrow[r, "s"] & Y \arrow[r, "u"] & C \arrow[r, "\delta"] & Z[-1]
\end{tikzcd}$$

we can also embed uf into an exact triangle

we furthermore get a morphism of triangles by a rotated (TR3).

Since $s \in S$, from the LES on H we get that $H^i(C) = 0$ $\forall i$.

Therefore we get that also $t \in S$.

The proof of the symmetric statement is similar.

(iii) If $f, g : X \to Y$ are so that there is some $s \in S$ with $sf = sg$, we again embed s into a triangle

$Z \xrightarrow{u} Y \xrightarrow{s} Y' \xrightarrow{\delta} Z[-1]$ and applying H we obtain $H^i(Z)=0$ $\forall i$. Let $h=f-g$, then consider the exact triangle

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "v"] & X \arrow[r] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] & X[-1] \arrow[d, "Tv"] \\ Z \arrow[r, "u"] & Y \arrow[r, "s"] & Y' \arrow[r] & Z[-1] \end{tikzcd}$$

we get v by TR3

so there is some $v: X \to Z$ s.t. $uv = f-g$, embed v into an exact triangle

$$X' \xrightarrow{t} X \xrightarrow{v} Z \xrightarrow{w} X'[-1]$$

since $\forall i$ $H^i(Z)=0$ we get $t \in S$ and furthermore $(f-g)t = \underbrace{uvt}_{0} = 0$ and hence $ft = gt$. The symmetric case is similar. We do not show 2. $\square$

<u>Corollary</u>: $\mathcal{D}(\mathcal{A})$, $\mathcal{D}^b(\mathcal{A})$, $\mathcal{D}^+(\mathcal{A})$, and $\mathcal{D}^-(\mathcal{A})$ are all triangulated categories.

<u>Theorem</u>: Assume that $\mathcal{A}$ has enough injectives (projectives) and that $\mathcal{D}(\mathcal{A})$ exists.

If Y is a bounded below (above) complex of injectives (projectives), then for every X, $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X,Y) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X,Y)$

$$(\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(Y,X) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(Y,X)).$$

$\hookrightarrow$ boundedness goes the other way than one would expect because we are working with cochain complexes

<u>Derived functors</u>

Consider some functor $F: \mathcal{A} \to \mathcal{B}$ of abelian categories, then we get a morphism of triangulated categories $\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ and we wish to extend it to a functor $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$. However, F need not send quasi-isomorphisms to isomorphisms so we cannot use the universal property of localization.

Solution: Kan extensions!

<u>Def</u>: The total **right** derived functor of $F: \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ a morphism of triangulated categories is the **left** Kan extension of F composed with the localization $q_{\mathcal{B}}: \mathcal{K}(\mathcal{B}) \to \mathcal{D}(\mathcal{B})$ along $q_{\mathcal{A}}: \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$:

$$\mathbf{R}F = \mathrm{Lan}_{q_{\mathcal{A}}}(F \circ q_{\mathcal{B}}): \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B}).$$

<u>Claim</u>: If F is exact, then the total left derived functor of F is the same as the one induced by the universal property of the localization.

<u>Proof</u>: Assuming F is exact $q_{\mathcal{B}} \circ F: \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ sends quasi-isos to isomorphisms. So there is a unique $\tilde{F}: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ s.t. $q_{\mathcal{B}} \circ F = \tilde{F} \circ q_{\mathcal{A}}$. Now assume we have some $G: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ with a natural transformation $\xi: q_{\mathcal{B}} \circ F \Rightarrow G \circ q_{\mathcal{A}}$.

$$\begin{tikzcd}
\mathcal{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "q_{\mathcal{A}}"'] \arrow[dr, phantom, "\Swarrow \xi"] & \mathcal{K}(\mathcal{B}) \arrow[d, "q_{\mathcal{B}}"] \\
\mathcal{D}(\mathcal{A}) \arrow[r, "\tilde{F}"'] \arrow[r, bend left, "G"] & \mathcal{D}(\mathcal{B})
\end{tikzcd}$$

we need that there is a unique $\eta: \tilde{F} \Rightarrow G$ s.t. $\xi = q_{\mathcal{B}} \eta q_{\mathcal{A}}$.

By the universal property of the localization there is a unique $G': \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ such that $G \circ q_{\mathcal{A}} = G'$. We can view ξ as a natural transformation thus $q_{\mathcal{B}} \circ F \Rightarrow G' (= G \circ q_{\mathcal{A}})$. Furthermore, by the 2-categorical nature of localization there is a unique $\eta: \tilde{F} \Rightarrow G$

corresponding to ξ which is a natural transformation between two homotopical functors $K(\mathcal{A}) \to D(\mathcal{B})$. $\square$

↳ I guess I am not proving the converse here, the Kan-extension actually being induced by the UP...

Remark: Essentially what is happening is that we are Kan-extending along a functor that is bijective on objects, so we need not worry about defining the natural transformation on extra objects, so if a functor factors through the localization, then it is its own left and right Kan extension.

Existence theorem: Let $F: K^+(\mathcal{A}) \to K^+(\mathcal{B})$ be a morphism of triangulated categories. If $\mathcal{A}$ has enough injectives, then the right derived functor $\mathbf{R}^+F$ exists on $D^+(\mathcal{A})$, and if I is a bounded below complex of injectives, then $\mathbf{R}^+F(I) \cong qF(I)$.

Dually, if $\mathcal{A}$ has enough projectives, then the left derived functor $\mathbf{L}^-F$ exists on $D^-(\mathcal{A})$ and if P is a bounded above chain complex of projectives, then $\mathbf{L}^-F(P) \cong qF(P)$.

Proof: This comes down to $K^+(\mathcal{I}) \subseteq K^+(\mathcal{A})$ being a localizing subcategory where $K^+(\mathcal{I})$ is the subcategory of bounded below complexes of injectives. Every quasi-isomorphism in $K^+(\mathcal{I})$ is an isomorphism (using the injective universal property one builds a chain homotopy, boundedness is crucial here) so $D^+(\mathcal{I}) \cong K^+(\mathcal{I})$. Given a complex $X \in K(\mathcal{A})$ we can take a right Cartan-Eilenberg resolution and get $X \to \mathrm{Tot}(I_{\cdot,\cdot})$ a quasi-isomorphism (check using s.seq. of a double complex first taking vertical

cohomology. By one of our earlier remarks then $K^+(\mathcal{I}) \cong D^+(\mathcal{A})$.
Since in $K^+(\mathcal{I})$ quasi-isos are really chain homotopy equivalences, F preserves those and so it is homotopical on $K^+(\mathcal{I})$. $\square$

<u>Corollary</u>: Let $F: \mathcal{A} \to \mathcal{B}$ be a functor of additive categories.

① If $\mathcal{A}$ has enough injectives, then the hyper-derived functors $\mathbb{R}^i F(X)$ give the cohomology of $\mathbf{R}F(X)$: $\forall i \quad \mathbb{R}^i F(X) \cong H^i \mathbf{R}^+ F(X)$

② If $\mathcal{A}$ has enough projectives, then the hyper-derived functors $\mathbb{L}^i F(X)$ give the cohomology of $\mathbf{L}F(X)$: $\forall i \quad \mathbb{L}^i F(X) \cong H^i \mathbf{L}^+ F(X)$

<u>Def</u>: Let $F: K(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. A complex X in $K(\mathcal{A})$ is called F-acyclic, if $\forall i \quad H^i(F(X)) = 0$.

<u>Generalized Existence Theorem</u>: Suppose, that $K' \subseteq K^+(\mathcal{A})$ is a triangulated subcategory of $K^+(\mathcal{A})$ such that

① Every $X \in K^+(\mathcal{A})$ admits a quasi-isomorphism $X \to X'$ to an object $X' \in K'$.

↳ in particular, if $s: X \to Y$ is a quasi-iso, then cone(s) is exact, so if $X, Y \in K'$ then cone$(s) \in K'$ is exact & this will mean F preserves quasi-isos of K'!

② Every exact complex in K' is F-acyclic.

Then $D' \cong D^+(\mathcal{A})$ and $\mathbf{R}F: D^+(\mathcal{A}) \cong D' \xrightarrow{\mathbf{R}'F} D^+(\mathcal{B})$ is a total right derived functor of F.

↳ in particular if ① holds then K' is a localizing subcategory using Lemma 10.3.13. (3), as a $C \to B'$ there can be chosen in S so the composite is guaranteed to be in S.

<u>The total tensor product</u>

Let R be a ring, $A, B \in Ch({}_R\mathrm{Mod})$ cochain complexes.

Def.: The total tensor product of $A, B \in K^-({}_R\mathrm{Mod})$ is

$$A \otimes_R^{\mathbb{L}} B = \mathbf{L}^- \mathrm{Tot}^{\oplus}(A \otimes_R -) B.$$

So the total left derived functor of $K^-({}_R\mathrm{Mod}) \longrightarrow K(Ab)$

$$B \longmapsto \mathrm{Tot}^{\oplus}(A \otimes_R B)$$

Lemma: If $A, A', B \in K^-({}_R\mathrm{Mod})$ and $A \to A'$ is a quasi-isomorphism, then $A \otimes_R^{\mathbb{L}} B \cong A' \otimes_R^{\mathbb{L}} B$.

Proof: We can change B up to quasi-isomorphism as $A \otimes_R^{\mathbb{L}} -$ is homotopical, therefore we may assume, that B is a complex of flat (projective) modules. In this case by the generalized existence theorem $A \otimes_R^{\mathbb{L}} B \cong \mathrm{Tot}^{\oplus}(A \otimes_R B)$ and $A' \otimes_R^{\mathbb{L}} B \cong \mathrm{Tot}^{\oplus}(A' \otimes_R B)$.
Now $A \to A'$ gives a comparison of double complexes which gives a comparison of spectral sequences of double complexes:

$$E_1^{p,q} \cong H^q(A) \otimes_R B^p \Rightarrow H^{p+q}(\mathrm{Tot}^{\oplus}(A \otimes_R B)) \quad \text{and}$$

$$E_1'^{p,q} \cong H^q(A') \otimes_R B^p \Rightarrow H^{p+q}(\mathrm{Tot}^{\oplus}(A' \otimes_R B))$$

since $A \to A'$ is a quasi-iso this comparison is an isomorphism, providing that $A \otimes_R^{\mathbb{L}} B \to A' \otimes_R^{\mathbb{L}} B$ is a quasi-isomorphism. $\square$

Theorem: The total tensor product is a functor

$$- \otimes_R^{\mathbb{L}} - : D^-({}_R\mathrm{Mod}) \times D^-({}_R\mathrm{Mod}) \longrightarrow D^-(Ab).$$

Its cohomology is the hypertor: $\mathbf{Tor}_i^R(A, B) \cong H^{-i}(A \otimes_R^{\mathbb{L}} B)$.

Corollary: If A and B are R-modules viewed as cochain complexes concentrated at degree zero, then $\mathrm{Tor}_i^R(A, B) \cong H^{-i}(A \otimes_R^{\mathbb{L}} B)$.

Sheaf cohomology

Let $X \in \mathrm{Top}$ and $\mathcal{F} \in Sh(X, Ab)$. $\Gamma : Sh(X, Ab) \longrightarrow Ab$, $\mathcal{F} \longmapsto \mathcal{F}(X)$ is the global sections functor. $Sh(X, Ab)$ has enough injectives and we can thus form $R^+\Gamma : D^+(X) \longrightarrow D^+(Ab)$. The usual sheaf cohomology is $H^i(X, \mathcal{F}) = H^i(R^+\Gamma(\mathcal{F}))$. where we take $\mathcal{F}$ to be concentrated in degree zero !

D (bounded below cochain complexes of sheaves)

A remark on this: injectives in $Sh(X, Ab)$ are as in any abelian category for existence of enough injectives see Stacks project 19.4.1. So on $Sh(X, Ab)$ we can define the (NOT total!) derived functor of Γ to get sheaf cohomology.

Ext & RHom

Let $A, B \in K(\mathcal{A})$. We have $\mathrm{Hom}_{K(\mathcal{A})}(A, B[-n]) \cong H^n(\mathrm{Hom}(A, B))$ because $\mathrm{Hom}(A, B)$ is the total complex of the double complex

$C_{p,q} := \mathrm{Hom}(A^{-p}, B^q)$ with differentials

$d^v(f)(a) = (-1)^{p+q+1} d^B(f(a)) \quad \forall a \in A^p$

we need to convert A to a chain complex to get a double complex

$$\begin{tikzcd}
\mathrm{Hom}(A^{-p+1}, B^{q+1}) \arrow[r] & \mathrm{Hom}(A^{-p}, B^{q+1}) \\
\mathrm{Hom}(A^{-p+1}, B^q) \arrow[u] \arrow[r, "(d^A)^* = d^h"'] & \mathrm{Hom}(A^{-p}, B^q) \arrow[u, "(-1)^{p+q+1}(d^B)_* = d^v"']
\end{tikzcd}$$

$\mathrm{Hom}(A^{-p+1}, B^q) = C_{p-1,q}$, $\mathrm{Hom}(A^{-p}, B^q) = C_{p,q}$

$d^h(f)(a) = f(d^A(a))$

an n-cycle is a sequence $\{f_p \in \mathrm{Hom}(A^{-p}, B^{n-p})\}_{p \in \mathbb{Z}} \in \mathrm{Tot}^\pi(\mathrm{Hom}(A, B))$

s.t. $\forall p \quad f_p d^A = (-1)^n f_{p+1} d^B$ so we get a chain map

$f : A \longrightarrow B[-n]$ $\qquad (B[-n])^p = B^{n+p}$!

a boundary is a null-homotopic chain map: there is some $\{s_p \in \mathrm{Hom}(A^{-p}, B^{n-p-1})\}_{p \in \mathbb{Z}}$ s.t. $(-1)^n s_p d^B + d^A s_{p-1} = f_p$!

Both $\mathrm{Hom}(A,-)$ and $\mathrm{Hom}(-,B)$ a morphisms of triangulated categories and Hom is a bimorphism

$$\mathrm{Hom}: K(\mathcal{A})^{op} \times K(\mathcal{A}) \longrightarrow K(Ab).$$

<u>Def.</u>: Let $A, B \in K(\mathcal{A})$. The n^{th} hyperext of A and B is the abelian group

$$\mathbf{Ext}^n(A,B) = \mathrm{Hom}_{D(\mathcal{A})}(A, T^n B).$$

We get cannonical maps

$$H^n \mathrm{Hom}(A,B) \cong \mathrm{Hom}_{K(\mathcal{A})}(A, T^n B) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(A, T^n B) = \mathbf{Ext}^n(A,B).$$

<u>Def.</u>: Suppose, that $\mathcal{A}$ has enough injectives, so that

$$\mathbf{R}^+ \mathrm{Hom}(A,-) : D^+(\mathcal{A}) \longrightarrow D(Ab) \quad \text{exists} \quad \forall A \in K(\mathcal{A}).$$

Assume that $B \to B'$ is a quasi-isomorphism and B' is a bounded below complex of injectives, then

$$H^n \mathbf{R}^+ \mathrm{Hom}(A,B') \cong H^n \mathrm{Hom}(A,B') \cong \mathrm{Hom}_{K(\mathcal{A})}(A,B') \cong \mathrm{Hom}_{D(\mathcal{A})}(A,B).$$

<u>Corollary</u>: If $\mathcal{A}$ has enough injectives, then for any $A, B \in \mathcal{A}$ viewed as chain complexes concentrated in degree zero $\mathbf{Ext}^n(A,B)$ gives the usual ext groups.

<u>Replacing spectral sequences</u>

We will replace spectral sequences by isomorphisms in the derived category.

Composition theorem: Let $K \subseteq K(\mathcal{A})$ and $K' \subseteq K(\mathcal{B})$ be localizing triangulated subcategories, and let $G: K \to K'$, and $F: K' \to K(\mathcal{C})$ be morphisms of triangulated categories. If $\mathbf{R}F$, $\mathbf{R}G$, and $\mathbf{R}(FG)$ exists and $\mathbf{R}F(\mathcal{D}) \subseteq \mathcal{D}'$, then

① there is a unique natural transformation $\zeta: \mathbf{R}(FG) \Rightarrow \mathbf{R}F \circ \mathbf{R}G$ such that the following diagram commutes in $\mathcal{D}(\mathcal{C})$ for all $A \in K$:

$$\begin{tikzcd} qFG(A) \arrow[r, "\xi_F"'] \arrow[d, "\xi_{F,G}"] \arrow[dr, phantom, "@"] & \mathbf{R}F(qG(A)) \arrow[d, "\xi_G"] \\ \mathbf{R}(FG)(qA) \arrow[r, "\zeta_{qA}"'] & (\mathbf{R}F)(\mathbf{R}G)(qA) \end{tikzcd}$$

where the ξ natural transformations are the ones in the structure of Kan-extensions

② suppose that there are triangulated subcategories $K_0 \subseteq K$ and $K_0' \subseteq K'$ satisfying the assumptions of the generalized existence theorem for G and F, and suppose that $G(K_0) \subseteq K_0'$. Then ζ is an isomorphism $\zeta: \mathbf{R}(FG) \cong (\mathbf{R}F)\circ(\mathbf{R}G)$.

Proof: ① is clear by the universal property of $\mathbf{R}(FG)$.

② for any $A \in K_0$,

$$\mathbf{R}(FG)(qA) = qFG(A) \cong \mathbf{R}F(qG(A)) \cong \mathbf{R}F(\mathbf{R}G(qA)). \square$$

Corollary: Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and let $G: \mathcal{A} \to \mathcal{B}$ and $F: \mathcal{B} \to \mathcal{C}$ be left exact functors. If G sends injective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$, then $\zeta: \mathbf{R}^+(FG) \cong (\mathbf{R}^+F)\circ(\mathbf{R}^+G)$ and there is a convergent spectral sequence for all $A \in D^+(\mathcal{A})$

$$E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

Proof: Recall the hypercohomology spectral sequence

$$(R^pF)(\underbrace{H^q(\mathbf{R}G(A))}_{R^qG(A)}) \Rightarrow \underbrace{R^{p+q}F(\mathbf{R}G(A))}_{H^{p+q}\mathbf{R}F\mathbf{R}G(A)} \qquad H^{p+q}\mathbf{R}F\mathbf{R}G(A) = H^{p+q}\mathbf{R}FG(A) =$$

$$= R^{p+q}(FG)(A)$$

because we use the composition theorem with

$\mathcal{L}_0 = K(\mathcal{I}) \subseteq \mathcal{K} = K(\mathcal{A})$ and $\mathcal{K}_0' = \mathcal{K}' = K(\mathcal{B})$! $\square$