Construction of the minimal resolution and basic computations

To do computations in the Adams spectral sequence we must first attempt to understands its E_2-page $\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$. We will set out to do this in a modest degree range in this talk.

Minimal resolution method

We want a resolution of $\mathbb{F}_2$ over $\mathcal{A}_2$

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{\varepsilon = \partial_0} P_0 \xleftarrow{\partial_1} P_1 \xleftarrow{\partial_2} P_2 \leftarrow \dots$$

such that
1. P_i are free over $\mathcal{A}_2$
2. $\ker(\partial_i) \subseteq \mathcal{A}_{2,+} \cdot P_i \quad \forall i \geq 0$

   where $\mathcal{A}_{2,+}$ is the ideal of all positive degree elements

   $\rightsquigarrow \mathcal{A}_{2,+}$ is the kernel of a map $\mathcal{A}_2 \to \mathbb{F}_2$

   $Sq^0 \longmapsto 1$

   $Sq^i \longmapsto 0 \quad \forall i > 0$
3. $(P_s)_t = 0 \quad \forall t < s$
4. $(P_t)_t \cong \mathbb{F}_2 \quad \forall t$

this is called minimal because it uses a minimal amount of generators for each P_i: let $x \in \ker(\partial_i)$ be expressed as $x = \sum_j a_j x_{i_j}$ where $\{x_{i_j}\}_j$ generate P_i as an $\mathcal{A}_2$-module; were $x \notin \mathcal{A}_{2,+} P_i$ so were there some k s.t. $a_{i_k} \in (\mathcal{A}_2)_0 \setminus \{0\}$, then we would have $0 = \partial_i(x) = \sum_j a_j \partial_i(x_{i_j})$ and would get $\partial_i(x_{i_k})$ expressed as a linear combination of other $\partial_i(x_{i_j})$'s meaning ∂_i would have the same image even without this generator rendering it superfluous!

Theorem: There exists a minimal resolution of $\mathbb{F}_2$ over $\mathcal{A}_2$ satisfying ①-④ with $P_0 = \mathcal{A}_2$, $P_1 = \bigoplus_{i \geq 0} \mathcal{A}_2[2^i]$ and $\partial_1 = \bigoplus_{i \geq 0} Sq^{2^i}$, and ∂_0 is the augmentation map.

Corollary: ⓐ If $s > t$, then $Ext^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = 0$

ⓑ For any $s \geq 0$, $Ext^{s,s}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$

ⓒ $Ext^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$

ⓓ $Ext^{1,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } t = 2^i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$

Proof: To calculate $Ext^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, we will use the minimal resolution provided to us by the theorem and apply $Hom_{\mathcal{A}_2}(-, \mathbb{F}_2[t])$.

For all $i, t \geq 0$, $Hom_{\mathcal{A}_2}(\partial_i, \mathbb{F}_2[t]) = 0$:

consider some $\varphi \in Hom_{\mathcal{A}_2}(P_{i-1}, \mathbb{F}_2[t])$ (with $P_{-1} = \mathbb{F}_2$, $\partial_{-1} = 0$)

Now $Hom_{\mathcal{A}_2}(\partial_i, \mathbb{F}_2[t])(\varphi) = \varphi \circ \partial_i$ and

$$im(\partial_i) = ker(\partial_{i-1}) \subseteq \mathcal{A}_{2,+} \cdot P_{i-1}$$

furthermore, $\varphi(\mathcal{A}_{2,+} \cdot P_{i-1}) \subseteq \mathcal{A}_{2,+} \cdot \underbrace{\mathbb{F}_2[t]}_{\substack{\cup\!\shortmid \\ \varphi(P_{i-1})}}$ because φ is an $\mathcal{A}_2$-module map.

But since acting by an element of $\mathcal{A}_{2,+}$ always raises the degree, and $\mathbb{F}_2[t]$ is concentrated in degree t, we get $\mathcal{A}_{2,+} \mathbb{F}_2[t] = 0$.

Therefore $Ext^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \cong Hom_{\mathcal{A}_2}(P_s, \mathbb{F}_2[t])$.

ⓐ follows directly from ③
ⓑ follows directly from ④
ⓒ $\mathrm{Ext}^{0,t}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Hom}_{\mathcal{A}_2}(\mathcal{A}_2, \mathbb{F}_2[t]) \cong \begin{cases} \mathbb{F}_2 & \text{if } t=0 \\ 0 & \text{otherwise} \end{cases}$

because $\mathcal{A}_2$ is the free $\mathcal{A}_2$ module generated by 1 and the map must be degree preserving, so it is entirely determined by where 1 is sent within $(\mathbb{F}_2[t])_0$.

ⓓ Similarly

$$\mathrm{Ext}^{1,t}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Hom}_{\mathcal{A}_2}\Big(\bigoplus_{i\geq 0} \mathcal{A}_2[2^i], \mathbb{F}_2[t]\Big) \cong \begin{cases} \mathbb{F}_2 & \text{if } t=2^i \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

<u>Proof of the theorem</u>: We build the minimal resolution satisfying ①-④ inductively.

Let $P_0 = \mathcal{A}_2$ and $\partial_0 : \mathcal{A}_2 \to \mathbb{F}_2$ the augmentation map.

$$Sq^i \longmapsto \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\ker(\partial_0) = \mathcal{A}_{2,+} = \mathcal{A}_{2,+}\mathcal{A}_2$ by definition. ③ is trivial ④ just says $(\mathcal{A}_2)_0 \cong \mathbb{F}_2$ which holds.

Now assume, that we already have $P_{s-1} \xrightarrow{\partial_{s-1}} P_{s-2} \to \dots \to P_0 \to \mathbb{F}_2 \to 0$ exact satisfying ① - ④.

Let $M = \ker(\partial_{s-1})$. $M/\mathcal{A}_{2,+}M$ is an $\mathbb{F}_2$-vector space and therefore it has a basis $\{[x_\alpha]\}_\alpha$. Let P_s be the free $\mathcal{A}_2$-module

generated by $\{y_\alpha\}_\alpha$ and $\partial_s(y_\alpha) = x_\alpha \ \forall \alpha$.
(So $P_s \cong \bigoplus_\alpha \mathcal{A}_2[|x_\alpha|]$.)

Now for exactness, we need to see that $\{x_\alpha\}$ generates M:
any $m \in M$ can be written as $m = \sum \varepsilon_\alpha x_\alpha + \sum a_i m_i$
where $\varepsilon_\alpha \in \mathbb{F}_2$, $a_i \in \mathcal{A}_{2,+}$
since $|a_i| > 0$ we have $|m_i| < |m|$
we can iterate this process and M having elements only in positive degrees guarantees we will end up with expressing m in terms of the x_α.

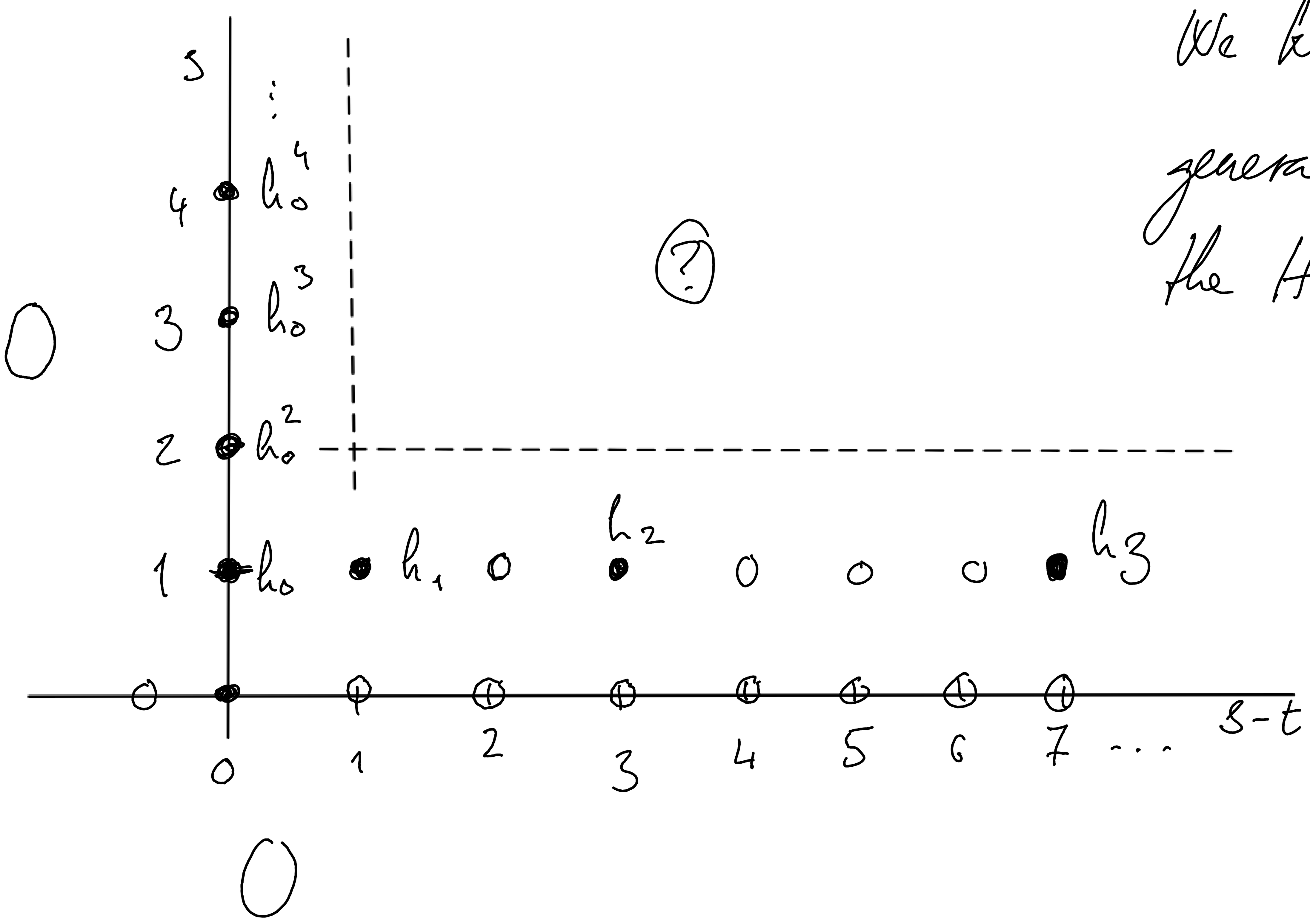
$\ker(\partial_s) \subseteq \mathcal{A}_{2,+} P_s$: let $\sum a_\alpha y_\alpha \in \ker(\partial_s)$,
if $A = \{\alpha \mid a_\alpha = 1 \in (\mathcal{A}_2)_0 \cong \mathbb{F}_2\}$ were not empty
then $\sum_{\alpha \in A} x_\alpha = \sum_{\alpha \notin A} a_\alpha x_\alpha$ and $\sum_{\alpha \in A} [x_\alpha] = 0$ would hold
$\{[x_\alpha]\}_\alpha$ being a basis this is impossible.

Since M is concentrated in degrees 1 higher than P_{s-1}, ③ is also inductively verified.

The proof of ④ we shall overlook, to see that $(P_s)_s$ is non-empty one must appeal to $\mathrm{Ext}^{1,1}_{\mathcal{A}_2}(H\mathbb{F}_2^*(H\mathbb{Z}), \mathbb{F}_2) \cong \mathbb{F}_2$ and use that in this degree the Hurewicz map $\mathbb{S} \to H\mathbb{Z}$ is an iso.

Note that $\mathcal{A}_{2,+}/\mathcal{A}_{2,+}\mathcal{A}_{2,+}$ is generated by Sq^{2^i} as these are exactly the indecomposable Steenrod squares as we have seen previously. $\square$

So argumentation using the minimal resolution of $\mathbb{F}_2$ gave us the E_2-page when $s=0$ or 1, or when $s-t \leq 0$.



We know that $h_0^i \in \mathrm{Ext}^{i,i}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2)$ generates, because we know that the Hurewicz map $\mathbb{S} \to H\mathbb{Z}$ induces $\mathrm{Ext}^{i,i}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow \mathrm{Ext}^{i,i}_{\mathcal{A}_2}(H\mathbb{F}_2^*(H\mathbb{Z}),\mathbb{F}_2)$ a multiplicative non-trivial map (being id on π_0) and we know the multiplicative structure on the right hand side.

Now a natural follow up question is whether we can say something about the survival of our newly discovered non-zero entries.

<u>Def.</u>: Let $[\alpha: \mathbb{S}^n \to \mathbb{S}] \in F^1(\pi_n^s)^\wedge_2$ and $x \in \mathrm{Ext}^{1,1+n}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2)$.

We say that x detects α whenever the composition

$$F^1(\pi^s_n)^\wedge_2 \twoheadrightarrow F^1(\pi^s_n)^\wedge_2 / F^2(\pi^s_n)^\wedge_2 \cong E_\infty^{1,1+n} \hookrightarrow E_2^{1,1+n} = \mathrm{Ext}^{1,1+n}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2)$$

sends α to x.

$$\mathrm{Ext}^{1,1+n}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } n=2^i-1 \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

<u>Proposition</u>: There is an element $\alpha \in \pi_{2^i-1}$ detected by $h_i \in E_2^{1,2^i}$ if and only if h_i survives to the E_∞-page.

Claim: α is detected by $h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ (the unique non-trivial element)

if and only if α has Hopf invariant 1.

Proof sketch:

Fact: α is mapped under this composition to the extension

$$0 \to \underbrace{H\mathbb{F}_2^*(\mathbb{S}^{n+1})}_{\mathbb{F}_2[2^i]} \to H\mathbb{F}_2^*(C(\alpha)) \to H\mathbb{F}_2^*(\mathbb{S}) \to 0$$

this comes from the LES of $\mathbb{S}^n \xrightarrow{\alpha} \mathbb{S} \to C(\alpha) \to \mathbb{S}^{n+1}$ using that

$\alpha \in F^1(\pi_n^{\mathbb{S}})^{\wedge}_2 \Rightarrow H\mathbb{F}_2^*(\alpha) = 0$:

Theorem: $f \in F_s[Y, X] \Leftrightarrow f$ can be written as a composite of s maps all of which are zero on $H\mathbb{F}_p^*$.

This SES of course splits over $\mathbb{F}_2$ so $H\mathbb{F}_2^*(C(\alpha))$ is concentrated in degrees 0 and 2^i.

The Sq^{2^i} action on it is non-trivial $\Leftrightarrow$ α has Hopf invariant 1.

This is exactly the scenario when this is not a split extension of $\mathcal{A}_2$-modules. $\square$

Corollary: The generators h_0, h_1, h_2, and h_3 survive to the E_∞-page and detect the Hopf maps $(\cdot 2) \in \pi_0^{\mathbb{S}}$, $\eta \in \pi_1^{\mathbb{S}}$, $\nu \in \pi_3^{\mathbb{S}}$, $\sigma \in \pi_7^{\mathbb{S}}$ respectively. $\square$