The analytic bootstrap at finite temperature

Analytic tools for thermal correlators

Thermal 2pt functions

Kinematics

$$g(\tau, x) = \langle \phi(\tau, x)\, \phi(0,0) \rangle_\beta \qquad (x = |\vec{x}|)$$

If $x = 0$: $\quad g(\tau) = g(\tau, 0)$

$$z = \tau + ix \qquad \bar{z} = \tau - ix$$

$$g(z, \bar{z}) = g(\tau, x)$$

$$r^2 = z\bar{z} \qquad w^2 = \frac{z}{\bar{z}}$$

OPE $\left(\sqrt{\tau^2 + \vec{x}^2} < \beta\right) \Rightarrow \quad g(z,\bar{z}) = \sum_{\mathcal{O}} a_{\mathcal{O}}\, f_{\Delta, J}(z, \bar{z})$

$$f_{\Delta, J}(z, \bar{z}) = (z\bar{z})^{\frac{\Delta}{2} - \Delta_\phi}\, C_J^{(\nu)}\left(\frac{z + \bar{z}}{2\sqrt{z\bar{z}}}\right) \qquad \nu = \frac{d-2}{2}$$

$$a_{\mathcal{O}} = \frac{J!}{2^J (\nu)_J} \frac{f_{\phi\phi\mathcal{O}}\, b_{\mathcal{O}}}{c_{\mathcal{O}}}$$

$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J} \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^\Delta} \left(e^{\mu_1} \cdots e^{\mu_J} - \text{traces} \right)$$

$$e^\mu = \delta^{0\mu}$$

$$g(\tau) = \frac{1}{\tau^{2\Delta_\phi}} \sum_\Delta a_\Delta \left(\frac{\tau}{\beta} \right)^\Delta$$

$$a_\Delta = \sum_{\mathcal{O},\, \Delta_{\mathcal{O}} = \Delta} a_{\mathcal{O}}\, C_J^{(\nu)}(1)$$

For the rest we set $\beta = 1$

Periodicity

KMS condition: $g(z, \bar{z}) = g(1 - \bar{z}, 1 - z)$

s-channel t-channel

Reality conditions: $g(z, \bar{z}) = g(\bar{z}, z) \longleftrightarrow g(r, w) = g(r, w^{-1})$

$g(z, \bar{z}) = g(-z, \bar{z}) \longleftrightarrow g(r, w) = g(r, -w)$

Reconstructing correlators at zero spatial separation

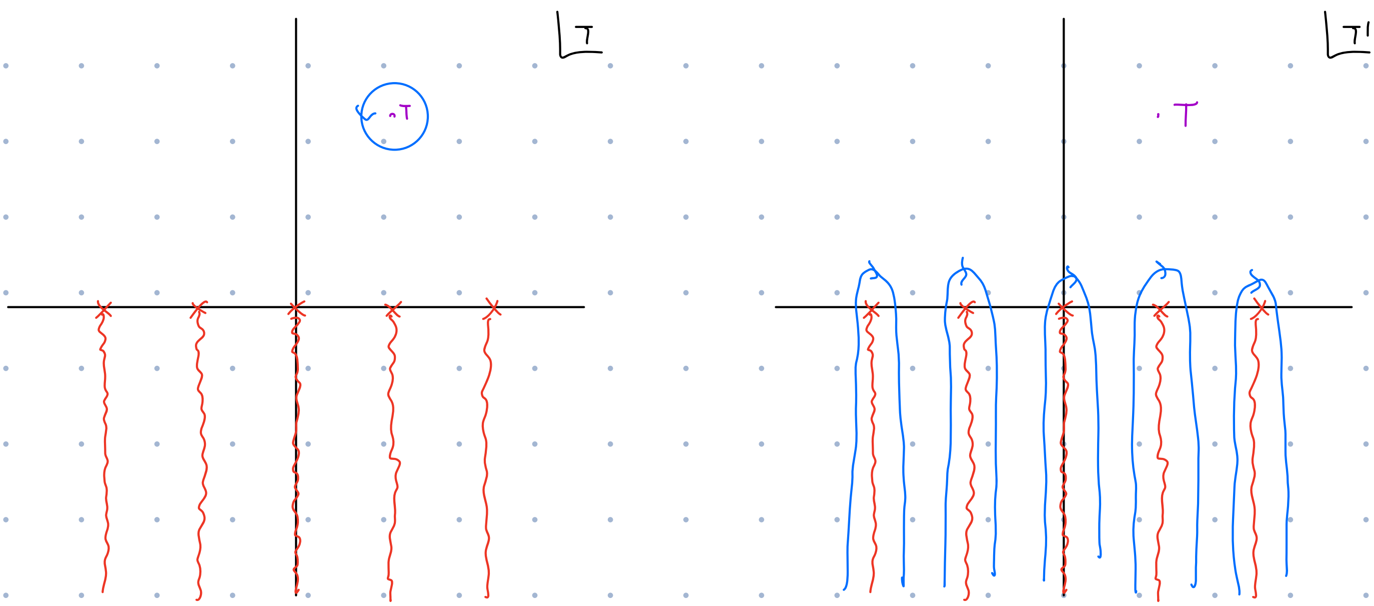
Dispersion relation

$g(\tau)$ as a function of $\tau \in \mathbb{C}$

simple poles in $\tau = n\beta$ $n \in \mathbb{Z}$

If branch cuts exist they lie along the imaginary direction and must extend to infinity

$$g(\tau) = \frac{1}{2\pi i} \oint_C d\tau' \, \frac{g(\tau')}{\tau' - \tau}$$



$|\tau| \to \infty$

$$g(\tau) = g_{dr}(\tau) + g_{arcs}(\tau)$$

$$g_{dr}(\tau) = \sum_{m=-\infty}^{\infty} \int_{-i\infty}^{0} \frac{d\tau'}{2\pi i} \frac{\mathrm{Disc}\, g(\tau')}{\tau' + m - \tau}$$

Arc contributions

$$g_{arcs}(\tau) = g_{arcs}(1-\tau)$$

$$g(\tau) = \langle \phi(\tau)\phi(0)\rangle_\beta = \langle \psi | e^{i\tau P_{KK}} | \psi \rangle$$

Kaluza-Klein compactification $\Big|$ $|\psi\rangle = \phi(0)|0\rangle_{\mathbb{R}^{d-2}\times S^1_\beta}$

V, U operators: $V = e^{-\mathrm{Im}(\tau) P_{KK}}$ $\quad U = e^{i\mathrm{Re}(\tau) P_{KK}}$

$$e^{i\tau P_{KK}} = V^{\frac{1}{2}} U V^{\frac{1}{2}}$$

U is unitary, V is positive Hermitian

$$|g(\tau)|^2 \leq \langle \psi | V^{\frac{1}{2}} V^{\frac{1}{2}} | \psi \rangle \langle \psi | V^{\frac{1}{2}} U^\dagger U V^{\frac{1}{2}} | \psi \rangle = \langle \psi | V | \psi \rangle^2$$

$$|g(\tau)| \leq |g(i\,\mathrm{Im}(\tau))|$$

$$|g_{arcs}(\tau)| < C e^{|\tau|} , \quad |\tau| \to \infty$$

$$g_{arcs}(\tau) = k \qquad k \in \mathbb{R}$$

function $f(w)$:

- $f(w)$ is an entire function
- $f(w) = f(w+1)$
- $|f(w)| < e^{|w|}$

$$h(w) = \frac{f(w) - f(0)}{\sin(\pi w)} \qquad h(w) \text{ is entire}$$

$h(w) = h(w+2\pi)$, $h(w)$ is bounded as $|h(w)| \to 0$ as $Im(w) \to \pm\infty$

Liouville's theorem $\Rightarrow h(w) = \lambda \in \mathbb{C}$

$$\left.\begin{array}{l} f(w) - f(0) = \lambda \sin(\pi w) \\ |f(w)| < e^{|w|} \end{array}\right\} \Rightarrow \lambda = 0$$

$$\Rightarrow \quad f(w) = f(0) = \text{const}$$

A formula for $g(\tau)$

$$\mathrm{Disc}_{\mathrm{Im}(\tau)<0}\, \tau^{\Delta-2\Delta_\phi} = \tau^{\Delta-2\Delta_\phi} e^{\frac{3}{2}\pi i(\Delta-2\Delta_\phi)}\left(1-e^{-2\pi i(\Delta-2\Delta_\phi)}\right)$$

$$g(\tau) = \sum_{\Delta} a_\Delta \left[\zeta_H(2\Delta_\phi-\Delta, \tau) + \zeta_H(2\Delta_\phi-\Delta, 1-\tau) \right] + k$$

ζ_H: Hurwitz ζ-function $\longrightarrow$ $\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$ $\quad$ $\mathrm{Re}(s)>1$, $a \neq 0,-1,-2,\dots$

Generalized Free Fields $\quad$ $g_{GFF}(\tau,\Delta) = \zeta_H(2\Delta,\tau) + \zeta_H(2\Delta,1-\tau)$

$$g(\tau) = \sum_{\Delta} a_\Delta\, g_{GFF}\left(\tau, \Delta_\phi - \frac{\Delta}{2}\right) + k$$

Any thermal 2pt function at zero spatial separation admits an expansion in terms of GFF correlators, with coeff given by the the thermal OPE data

g_{GFF} vanishes when $\Delta - 2\Delta_\phi \in 2\mathbb{Z}_{>0}$

Double twist operators do not contribute in a GFF theory

Classical result from the analytic conformal bootstrap:

$$\Delta = 2\Delta_\phi + 2n + J + \gamma \qquad \gamma \sim O\left(\frac{1}{J}\right)$$

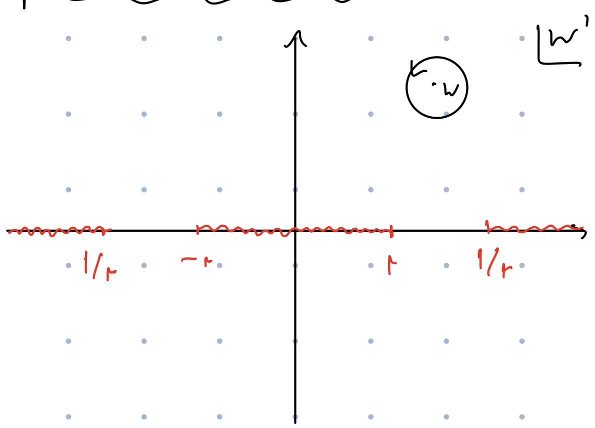
as $J \to \infty$

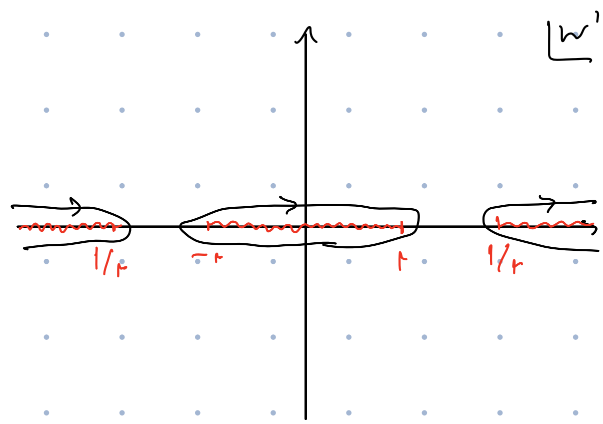
At the large spin limit:

The OPE spectrum approaches

Reconstructing correlators at non-zero spatial separation

Dispersion relation





$$g(r,w) = \oint_{\mathcal{C}} \frac{dw'}{2\pi i} \frac{g(r,w')}{w'-w}$$

$$g_{dr}(r,w) = \int_0^r \frac{dw'}{2\pi i} \cdot \frac{w^2(1-w'^4)}{w'(w'-w)(w'+w)(1-w^2w'^2)} \,\mathrm{Disc}\, g(r,w')$$

$$g(r,w) = g_{dr}(r,w) + g_{arcs}(r,w)$$

$|w| \to \infty$

OPE inversion and consistency conditions

$$\left.\begin{aligned} g(z,\bar z) &= \sum_{\mathcal{O}} a_{\mathcal{O}} f_{\Delta,J}(z,\bar z) \\ g_{dr}(r,w) &= \int_0^r \frac{dw'}{2\pi i} \cdot \frac{w^2(1-w'^4)}{w'(w'-w)(w'+w)(1-w^2w'^2)} \,\mathrm{Disc}\, g(r,w') \end{aligned}\right\}$$

$$\underset{\mathrm{Re}(\bar z)>0}{\mathrm{Disc}}\, g(r,w') = \sum_{\mathcal{O}} a_{\mathcal{O}_e} \underset{\mathrm{Re}(\bar z)>0}{\mathrm{Disc}}\, f_{\Delta,J}(z,\bar z)$$

$\Delta - 2\Delta_\phi \in 2\mathbb{Z}^{>0}$ blocks don't contribute because:

$$f_{\Delta,J}(1-z,1-\bar{z}) \sim \sum_{\Delta<2\Delta_\phi} \frac{1}{(1-\bar{z})^{\Delta_\phi-\Delta/2}} + \text{regular}$$

$$\underset{Re(\bar{z})>0}{\text{Disc}} (1-\bar{z})^{\alpha} = 0 \qquad \text{for } \alpha \in \mathbb{Z}^{>0}$$

$$\underset{Re(\bar{z})>0}{\text{Disc}} (1-\bar{z})^{\alpha} = 2i\sin(\alpha\pi)(\bar{z}-1)^{\alpha}\Theta(Re(\bar{z})-1) \qquad \text{for } \alpha \notin \mathbb{Z}^{>0}$$

At zero T the procedure reproduces g_{dr}

But at finite T the OPE doesn't converge everywhere

$$\left.\begin{array}{l} \text{In the integral } w \in (0,r) \\ r = \sqrt{x^2+\tau^2} \end{array}\right\} \Rightarrow r < 1$$

$g_{dr}(r,w)$ does not satisfy the KMS condition

Lorentzian inv formula $\Rightarrow$ $a(\Delta,J)$ $\Rightarrow$ poles $\downarrow$ OPE coeff

- OPE contributions (polynomially suppressed in J)

$$\int_1^\infty \frac{d\bar{z}}{\bar{z}}\, \bar{z}^{\Delta_\phi - \bar{h} - m}\, \mathrm{Disc}(1-\bar{z})^c \sim J^{-c-1} + \dots$$

$$\bar{h} = \frac{1}{2}(\Delta + J), \quad m \in \mathbb{Z}^{\geq 0}$$

- Out-of-OPE contributions: (exp suppressed in J)

$$\int_2^\infty \frac{d\bar{z}}{\bar{z}}\, \bar{z}^{\Delta_\phi - \bar{h} - m} f(\bar{z}) \sim 2^{-J} + \dots$$

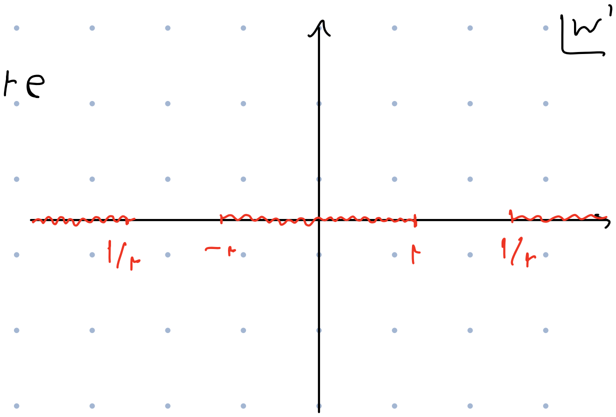
These terms are missing from the Lorentzian inv formula and they are not captured by the dispersion relation

$$g(z,\bar{z}) = g_{OPE}(z,\bar{z}) + g_{out\text{-}of\text{-}OPE}(z,\bar{z})$$

Consistency conditions to reconstruct $g_{\text{out-of-OPE}}(z,\bar{z})$

1) KMS condition

2) Analytic structure



3) Consistency with the OPE

The dispersion relation is expected to reproduce the correct OPE spectrum up to low-spin operators

↪ encoded in g_{arcs}

4) Regge boundedness

In the Regge limit: $w \to \infty$, r fixed

$g(r,w)$ is polynomially bounded

5) Clustering at large distances:

$$g(z,\bar{z}) \xrightarrow{x\to\infty} \langle\phi\rangle_\beta^2$$

A numerical approach

$$a_\mathcal{O} = a_\mathcal{O}^{(dr)}\left(\underbrace{1}_{g_{OPE}} + \underbrace{\sum_{n=1}^{\infty}\frac{c_n}{n^J}}_{g_{\text{out-of-OPE}}}\right)$$

c_n can be fixed numerically from the KMS condition:

$$g_{cand}(z,\bar{z}) \overset{!}{=} g_{cand}(1-z,1-\bar{z})$$

(need to truncate the sum $\to n_{max}$)

The generalized method of images

Impose KMS explicitly

any non-periodic function $f(\tau)$ can be used to built a periodic function:

$$\tilde{f}(\tau) = \sum_{m=-\infty}^{\infty} f(\tau - m)$$

$$\Rightarrow \quad g(z,\bar{z}) = \frac{1}{2} \sum_{m=-\infty}^{\infty} g_{dr}(z-m, \bar{z}-m) + g_{arcs}(z,\bar{z})$$

"generalized method of images"

satisfies consistency conditions 1-4,
5 needs to be on a case-by-case basis

Arc contributions

low spin operators $J < J^*$

$$g_{arcs}(\tau, x) = \sum_{\Delta} \sum_{J<J^*} a_{\mathcal{O}}^{(arcs)} (x^2+\tau^2)^{\frac{\Delta}{2}-\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{x^2+\tau^2}}\right) =$$

$$= \sum_{J<J_*} F_J(\tau, x)\, C_J^{(\nu)}\left(\frac{\tau}{\sqrt{x^2+\tau^2}}\right)$$

If the generalized method of images reproduces the correct clustering decomposition then:

$$\lim_{x\to\infty} g_{arcs}(\tau, x) = 0 \quad \Rightarrow \quad \lim_{x\to\infty} F_J(\tau, x) = 0$$

but using KMS this leads to

$$F_J = 0$$

$\Rightarrow$ the generalized method of images does not reproduce the correct clustering decomposition

$\Rightarrow$ g_{arcs} is theory-dependent