Notes by Talia

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Random Groups Final Lecture

Q. Are random groups residually finite?

$\forall g \in G \setminus \{1\}, \exists H$ fin, $\varphi : G \twoheadrightarrow H$ s.t. $\varphi(g) \neq 1$.

→ Alessandro doesn't think hyperbolic groups are residually finite.

Fact: G random $= F_m / R$ (relators of length ℓ)
Fix H finite group. If $\ell >> 0$, then $G \not\twoheadrightarrow H$.

Temperature model

· Define ν on $P = \{$all presentations on m generators$\}$
· Then pick g according to ν.

$G = F_m / R$, $d \in \mathbb{R}$

$\forall w \in F_m$,
$$\mathbb{P}_d\{w \in R\} = c(2m-1)^{(d-1)|w|}$$

Fix ℓ, $\mathbb{E}\left\{ \begin{array}{c} \# w \in R \\ |w| = \ell \end{array} \right\} = \underbrace{c(2m-1)^{(d-1)\ell}}_{P(w \in R)} \cdot \underbrace{2m(2m-1)^{\ell}}_{\# \text{ of words } w \text{ of length } \ell}$

$$R = \bigcup_{\ell=0}^{+\infty} R_\ell$$ ← "all lengths density model"

Temperature: $\mathbb{P}$(particle of "energy" t) $\sim e^{-t}$

"Energy" of a word $= |w|$

$$T = \frac{1}{(1-d)\log(2m-1)}$$

Fact: If $d > 1/2$, then $\mathbb{P}_d(G = \{1\}) = 1$.

Fact: $\forall d$, $\mathbb{P}_d(G = \{1\}) > 0$.

Hyp. groups are fin. presented, but R in this way gives infinitely presented groups ∵
Cannot hope for phase transition w/ hyperbolic groups.

Conj. If $d < 1/2$, then there is a +ve probability that $G = F_m/R$ is a direct limit of infinite hyperbolic groups of geom. dim. 2.

Thm: [Gromov-Cartan-Hadamard]

$G = \langle a_1, \dots, a_m \mid r_1, \dots, r_k \rangle$ $\qquad \rho = \dfrac{\max |r_i|}{\min |r_j|}$.

$\forall C, \exists T(c,\rho)$ s.t. the following holds:

If: $\forall D$ vKD with $< T$ cells, $|\partial D| \geq C|D|$,

Then: $\forall D$, $|\partial D| \geq (C-1)|D|$

* Finds hyperbolicity constant ∵

What happens when density is < 0?

$$\mathbb{E}[|R_\ell|] = (2m-1)^{d\ell}$$

$$\sum_{\ell=0}^{+\infty} P(\exists w \in R \mid |w| = \ell) = \sum_{\ell=0}^{+\infty} (2m-1)^{d\ell} < 0.$$

Thm: (Borel-Canteli)

Collection of events E_i

$\sum_i \mathbb{P}(E_i) < \infty \Rightarrow$ only finitely many E_i are true at the same time.

P_d (length of relators is bounded) $= 1$.

$\Rightarrow$ $d < 0$, $\mathbb{P}_d(G$ is hyp, torsion-free$) > 0$.

$0 < d < 1/2$, $G = F_m / R$

Fact 1: G^{ab} is trivial $\Rightarrow$ G not abelian

Fact 2: G has no finite quotients (in the range $0 < d < 1/2$)

(Notes incomplete)