The Adams spectral sequence and $H\mathbb{Z}$

Let X spectrum that is bounded below and of finite type.

Let $\cdots \to X_2 \to X_1 \to X_0 = X$ be a Adams resolution, this induces an

.) Adams spectral sequence with

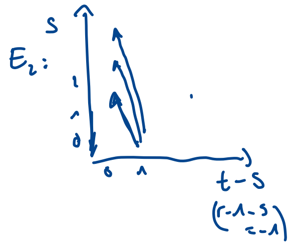
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(H^*(X,\mathbb{F}_p),\mathbb{F}_p) \Rightarrow \pi_{t-s}X^\wedge_p = \pi_{t-s}(X^\wedge_p)$$

with differentials on r-th page of bidegree $(r, r-1)$

there exists a filtration $(F^s\pi_*X)_{s\geq 0}$ on π_*X

by $F^s\pi_*X = \mathrm{im}(\pi_*X_s \to \pi_*X)$ s.t.

$$F^s\pi_n X / F^{s+1}\pi_n X \cong E_\infty^{s,n+s}.$$



.) $(F^s\pi_*X)_{s\geq0}$ is equivalent to the p-adic filtration on π_*X.

$A \supseteq pA \supseteq p^2A \dots$

F equiv. to G if

$\forall s\ \exists s' : G^{s'} \subseteq F^s$ and $\forall s\ \exists s'$ s.t. $F^{s'} \subseteq G^s$

last lecture

p-completion

$$A_p^\wedge = \lim_{\substack{\longleftarrow \\ n}} (\cdots \to A/p^nA \to A/p^{n-1}A \to \cdots \to A)$$

$$\mathbb{Z}_p^\wedge = \lim_{\longleftarrow} \mathbb{Z}/p^n\mathbb{Z} \quad \text{p-adic integers}$$

$$(\mathbb{Z}/q^l\mathbb{Z})_p^\wedge = \lim_{\substack{\longleftarrow \\ n}}\left(\cdots \to \mathbb{Z}/q^l\mathbb{Z}\big/p^n(\mathbb{Z}/q^l\mathbb{Z}) \to \mathbb{Z}/q^l\mathbb{Z}\big/p^n(\mathbb{Z}/q^l\mathbb{Z}) \to \cdots \to \right)$$

$$\mathbb{Z}/q^l\mathbb{Z} \xrightarrow{\cdot p^n} \mathbb{Z}/q^l\mathbb{Z} \begin{cases} \text{iso} & \text{if } (q,p)=1 \\ 0 & \text{else} \end{cases}$$

Now: Let $A = \bigoplus_\alpha \mathbb{Z} \oplus \bigoplus_{l=1}^{S} \mathbb{Z}/p^{k_l}\mathbb{Z} \oplus$ non-p-torsion

then $A_p^\wedge = \bigoplus_\alpha \mathbb{Z}_p^\wedge \oplus \bigoplus_{l=1}^{S} \mathbb{Z}/p^{k_l}\mathbb{Z}$

So if $\pi_*(A)$ has a free part or non p torsion then the filtration might not be complete.

.) $M = F^0 \subseteq F^1 \subseteq F^2 \subseteq F^3$

$\hat{M} = \lim_n M/F^n$

$\hat{F}^i = \lim_k F^i/F^{i+k}$

Funktor $0 \to A \to B \to C \to 0$ and $\Rightarrow H(B) \to H(C) \to H(A)[1]$ fibre sequence

$\Rightarrow H(A) \to H(B) \to H(C)$ fibre sequence

$\Rightarrow \pi_n(HA) \to \pi_n(HB) \to \pi_n(HC) \to \pi_{n-1}(HA)$ l.e.s.

$H(A) \otimes H\mathbb{F}_p \to H(B) \otimes H\mathbb{F}_p \dots$

dann π nehmen = homologie

$\dots \to H_n(HA, \mathbb{F}_p) \to H_n(HB, \mathbb{F}_p) \to H_n(HC, \mathbb{F}_p) \to H_{n-1}(HA, \mathbb{F}_p) \to \dots$ l.e.s.

$\mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_p$

$H(\mathbb{Z}) \to H\mathbb{Z} \to H\mathbb{F}_p$

<u>Example</u>: Eilenberg-Maclane Spectrum $H\mathbb{Z}$

↳ characterised by $\pi_0 H\mathbb{Z} = \mathbb{Z}$ and $\pi_i H\mathbb{Z} = 0$ $i \neq 0$

has the Adams resolution:

$$\begin{tikzcd}
\dots \arrow[r] & \overset{2}{H\mathbb{Z}} \arrow[r, "\cdot p"] & \overset{1}{H\mathbb{Z}} \arrow[r, "\cdot p"] \arrow[d] & \overset{0}{H\mathbb{Z}} \arrow[d] \\
 & & H\mathbb{F}_p \arrow[ul, dashed, "+1"] & H\mathbb{F}_p \arrow[ul, dashed, "+1"]
\end{tikzcd}$$

$H\mathbb{F}_p \xrightarrow{(*)} H\mathbb{Z}[1] \to H\mathbb{F}_p[1]$

$\left(\mathrm{cofib}\left(\overset{s+1}{H\mathbb{Z}} \xrightarrow{\cdot p} \overset{s}{H\mathbb{Z}} \right) \cong H\mathbb{F}_p \;,\; \underset{\mathcal{A}_p^*}{\underset{\|}{H^*(H\mathbb{F}_p, \mathbb{F}_p)}} \twoheadrightarrow H^*(\overset{s}{H\mathbb{Z}}, \mathbb{F}_p) \xrightarrow{\cdot p = 0} H^*(\overset{s+1}{H\mathbb{Z}}, \mathbb{F}_p) \right)$

$\mathbb{F}_p$ modules

this gives proj. resolution:

l.e.s.

$$\dots \to \overset{2}{\mathcal{A}_p^*[2]} \to \underset{H^*(H\mathbb{F}_p, \mathbb{F}_p)[1]}{\underset{\|}{\overset{1}{\mathcal{A}_p^*[1]}}} \xrightarrow{f} \underset{H^*(H\mathbb{F}_p, \mathbb{F}_p)}{\underset{\|}{\overset{0}{\mathcal{A}_p^*}}} \to \underset{\mathrm{coker}(f)}{\underset{\|}{H^*(H\mathbb{Z}; \mathbb{F}_p)}} \to 0$$

$H\mathbb{F}_p^*(H\mathbb{Z}) = \pi_{-n}(\mathrm{Hom}(H\mathbb{Z}, H\mathbb{F}_p)) = \pi_0(\mathrm{Hom}(H\mathbb{Z}, H\mathbb{F}_p[n])) = [H\mathbb{Z}, H\mathbb{F}_p[n]]$

$1 \in \mathcal{A}_p^*[1] = \mathcal{A}_p^*$

<u>maps</u>: Consider

$$\begin{tikzcd}
H\mathbb{Z} \arrow[r, "\cdot p"] \arrow[d, "pr."] & H\mathbb{Z} \arrow[r, "pr."] \arrow[d, "pr."] & H\mathbb{F}_p \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "*"] & H\mathbb{Z}[1] \arrow[d, "pr."] \\
H\mathbb{F}_p \arrow[r, "\cdot p"] & H\mathbb{Z}/p^2 \arrow[r, "pr."] & H\mathbb{F}_p \arrow[r, "\beta"] & H\mathbb{F}_p[1]
\end{tikzcd}$$

first

β bockstein

$\Rightarrow f$ is given by rigth multp. with the bockstein

$\Rightarrow H^*(H\mathbb{Z}, \mathbb{F}_p) = \mathcal{A}_p^* / \mathcal{A}_p^*[1] \cdot \beta$

ASS of $H\mathbb{Z}$:

$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(H^*(H\mathbb{Z}, \mathbb{F}_p), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & s = t \geq 0 \\ 0 & \text{else} \end{cases}$$

$\uparrow$ $\mathcal{X}$ proj. resolution of that

$$\mathrm{Hom}^*(\mathcal{X}, \mathbb{F}_p) = \mathrm{Hom}^*_{\mathcal{A}_p^*}(\mathcal{A}_p^*, \mathbb{F}_p) \to \mathrm{Hom}^*_{\mathcal{A}_p^*}(\mathcal{A}_p^*[1], \mathbb{F}_p) \to \dots$$

$\uparrow$ 1 lebt in grad 1

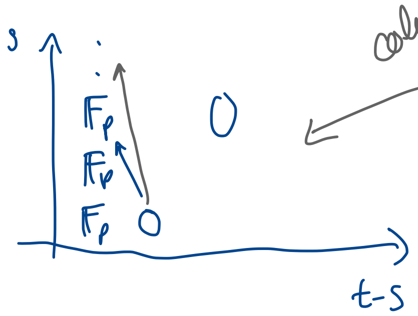
graded module over $\mathcal{A}_p^*$ nur in grad 0 $\mathbb{F}_p$

„calculate hom degree wise"

$$= \begin{array}{lllll} 0 \to 0 \to \mathbb{F}_p \to 0 & \mathrm{Hom}^2 \\ 0 \to \mathbb{F}_p \to \mathbb{F}_p \to 0 & \mathrm{Hom}^1 \\ \mathbb{F}_p \to 0 \to 0 \to 0 & \mathrm{Hom}^0 \end{array} = \mathbb{F}_p \to \mathbb{F}_p[+1] \to \mathbb{F}_p[+2] \to \dots$$

cohomologie

$E_\infty = E_2$



filtration:

$$F^s\pi_*(H\mathbb{Z}) = \mathrm{im}\left(\pi_*(H^s\mathbb{Z}) \xrightarrow{p^s} \pi_*(H^0\mathbb{Z})\right) = p^s \cdot \mathbb{Z}$$

$\mathbb{Z} \supseteq p\mathbb{Z} \supseteq p^2\mathbb{Z} \supseteq p^3\mathbb{Z} \dots$ with subquotients $p^s\mathbb{Z}/p^{s+1}\mathbb{Z} = \mathbb{F}_p \cong E_\infty^{s,s}$

(p-vervollständigung:

$$\pi_n(H\mathbb{Z})^\wedge_p = \begin{cases} \mathbb{Z}_p & n = 0 \\ 0 & \text{else} \end{cases}$$

$\mathbb{Z}_p \supseteq p \cdot \mathbb{Z}_p \supseteq p^2\mathbb{Z}_p \supseteq \dots$

$p^n\mathbb{Z}_p / p^{n+1}\mathbb{Z}_p \cong \mathbb{F}_p \cdot [p^n]$.)

pairing of Adams s.s

Let X, Y be spectra - bounded below and of finite type.

·) There exists a pairing : $E_r^{s,t}(X) \otimes_{\mathbb{F}_p} E_r^{s',t'}(Y) \longrightarrow E_r^{s+s',t+t'}(X \wedge Y)$

s.t. $d_r(x \cdot y) = (d_r x) \cdot y \pm x \cdot d_r y$

(on E_∞ it comes from : ·) graded ring $\pi_* X$ (multiplication on)

·) and $F_n \pi_\ell \cdot F_m \pi_k \subseteq F_{n+m} \pi_{\ell+k}$
the filtration is multiplicative

$$\begin{array}{ccccc} E_\infty & \otimes & E_\infty & \longrightarrow & E_\infty \\ \cong \| & & & & \\ F_s\pi_{t-s}/F_{s+1}\pi_{t-s} & \otimes & F_{s'}\pi_{t'-s'}/F_{s'+1}\pi_{t'-s'} & \longrightarrow & F_{s+s'}\pi_{t+t'-s-s'}/F_{s+s'+1}\pi_{t+t'-s-s'} \end{array}$$

·) Explicitly on E_2 :

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(\underbrace{H^*(X,\mathbb{F}_p)}_{M},\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{Ext}^{s',t'}_{\mathcal{A}_p^*}(\underbrace{H^*(Y,\mathbb{F}_p)}_{M'},\mathbb{F}_p)$$

$$\downarrow 1)$$

$$\mathrm{Ext}^{s+s',t+t'}_{\mathcal{A}_p^*\otimes_{\mathbb{F}_p}\mathcal{A}_p^*}(H^*(X,\mathbb{F}_p)\otimes_{\mathbb{F}_p} H^*(Y,\mathbb{F}_p),\mathbb{F}_p)$$

$$\downarrow 2) \quad \text{Künneth}$$

$$\mathrm{Ext}^{s+s',t+t'}_{\mathcal{A}_p^*}(H^*(X\wedge Y;\mathbb{F}_p),\mathbb{F}_p)$$

1) $$\bigoplus_{p+q=n} \mathrm{Ext}^{p,j}_{\mathcal{A}_p^*}(M,\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{Ext}^{q,j'}_{\mathcal{A}_p^*}(M',\mathbb{F}_p)$$

$$= \bigoplus_{p+q=n} H^p(\mathrm{Hom}_{\mathcal{A}_p^*}(P_\bullet,\mathbb{F}_p[j])) \otimes_{\mathbb{F}_p} H^p(\mathrm{Hom}_{\mathcal{A}^*}(P'_\bullet,\mathbb{F}_p[j']))$$

$$\xrightarrow{\text{Künneth}} H^n(\mathrm{Hom}_{\mathcal{A}_p^*}(P_\bullet,\mathbb{F}_p[j]) \otimes_{\mathbb{F}_p} \mathrm{Hom}_{\mathcal{A}_p^*}(P'_\bullet,\mathbb{F}_p[j']))$$

$$\xrightarrow{*} H^n(\mathrm{Hom}_{\mathcal{A}_p^*\otimes_{\mathbb{F}_p}\mathcal{A}_p^*}(P_\bullet\otimes_{\mathbb{F}_p}P'_\bullet,\mathbb{F}_p[j+j'])) = \mathrm{Ext}^{n,j+j'}_{\mathcal{A}_p^*\otimes_{\mathbb{F}_p}\mathcal{A}_p^*}(M\otimes_{\mathbb{F}_p}M',\mathbb{F}_p)$$

* M, N A module

·) $\mathrm{Hom}_A(M,N)\otimes_k \mathrm{Hom}_{A'}(M',N') \longrightarrow \mathrm{Hom}_{A\otimes_k A'}(M\otimes_k M', N\otimes_k N')$

$f\otimes f' \longmapsto (f\otimes f' : m\otimes m' \longmapsto f(m)\otimes f'(m'))$

2) ·) $\Delta: \mathcal{A}_p^* \longrightarrow \mathcal{A}_p^* \otimes_{\mathbb{F}_p} \mathcal{A}_p^*$ comultiplication on $\mathcal{A}_p^*$

pullback $\mathcal{A}$ module to $\mathcal{A}_p^*$ module

·) Künneth

For $X = Y$ a ring spectrum we have map $X \wedge X \longrightarrow X$

$\Rightarrow$ get multiplication

·) For map of multiplicative ring spectra $f: X \longrightarrow Y$ we get

morphism of mult. spectral sequences

<u>functorality:</u>

·) ASS is functorial in X

$\mathbb{S} \longrightarrow H\mathbb{Z}$ iso on π_0