FURTHER COMPUTATION USING THE MINIMAL RESOLUTION AND STABLE HOMOTOPY GROUPS

In the first part of the talk we will sketch the computation of $\mathrm{Ext}^{s,t}_{\mathcal{A}_2^*}(\mathbb{F}_2,\mathbb{F}_2)$ for $s,t\le 6$. For brevity, we write $\boxed{\mathcal{A} := \mathcal{A}_2^*}$.

Def. Let M be a graded module and $t\ge 0$. The <u>t-th truncation</u> is the graded module M_t with

$$\boxed{(M_{\le t})_n := \begin{cases} M_n & ; \; n\le t \\ 0 & ; \; \text{else} \end{cases}}.$$

It is immediate that for a graded $\mathcal{A}$-module M, $t\ge 0$,

$$\boxed{\mathrm{Hom}_{\mathcal{A}}(M,\mathbb{F}_2[t]) \cong \mathrm{Hom}_{\mathcal{A}}(M_{\le t},\mathbb{F}_2[t])}.$$

We will make use of this property for computation of the lower degree terms of $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$.

$\hookrightarrow$ We can truncate the minimal resolution.

Let

$$0 \leftarrow \mathbb{F}_2 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

be a minimal resolution.

Since we are only interested in $0\le s,t\le 6$, we only have to understand

$$\boxed{0 \leftarrow \mathbb{F}_2 \xleftarrow{\varepsilon} (P_0)_{\le 6} \xleftarrow{\partial_1} (P_1)_{\le 6} \leftarrow \cdots \xleftarrow{\partial_6} (P_6)_{\le 6} \leftarrow 0}$$

The advantage is that now we can compute the above sequence explicitly.

<u>Notation</u>: For X a set, let $\mathcal{A}X$ denote the free $\mathcal{A}$-module w/ gen. X.

- <u>$P_0 = \mathcal{A}$</u>, $\quad \varepsilon: \mathcal{A}_{\le 6} \to \mathbb{F}_2$

$$\ker\varepsilon = \langle Sq^1, Sq^2, Sq^4\rangle_{\mathcal{A}}.$$

$$\Rightarrow \quad 0 \leftarrow \mathbb{F}_2 \xleftarrow{\varepsilon} \mathcal{A}_{\le 6} \xleftarrow{\partial_1} \underset{\substack{\| \\ P_1}}{\mathcal{A}\{h_0,h_1,h_2\}_{\le 6}} \quad \text{w/ } |h_i| = 2^i \;\&\; \partial_1(h_i) = Sq^{2^i}.$$

- To compute the $\ker \partial_1$, we list the additive generators for both $\ker \varepsilon$ and $\mathcal{A}\{h_0, h_1, h_2\}$ and then compute the images of these w.r.t. the add. gens. of $\ker \varepsilon$. (+ use Adem rels.)

With some work we get

grad.	$\ker \partial_1$
1	/
2	$Sq^1 h_0$
3	/
4	$Sq^2 Sq^1 h_0$ $\quad$ $Sq^2 h_1 + Sq^3 h_0$
5	$Sq^3 Sq^1 h_0$ $\quad$ $Sq^3 h_1$ $\quad$ $Sq^4 h_0 + Sq^2 Sq^1 h_1 + Sq^1 h_2$
6	$Sq^4 Sq^1 h_0$ $\quad$ $Sq^5 h_0 + Sq^3 Sq^1 h_1$

def. $\overline{g_0} := Sq^1 h_0$, $\overline{g_1} := Sq^2 h_1 + Sq^3 h_0$, $\overline{g_2} := Sq^4 h_0 + Sq^2 Sq^1 h_1 + Sq^1 h_2$

Using Adem relations, we check that

$$\boxed{\ker \partial_1 = \langle \overline{g_0}, \overline{g_1}, \overline{g_2} \rangle_{\mathcal{A}}}.$$

$\Rightarrow$ Get next step in resolution:

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{\varepsilon} \mathcal{A}_{\leq 6} \xleftarrow{\partial_1} \mathcal{A}\{h_0, h_1, h_2\}_{\leq 6} \xleftarrow{\partial_2} \underbrace{\mathcal{A}\{g_0, g_1, g_2\}_{\leq 6}}_{=:\, P_2}$$

where $\partial_2(g_i) := \overline{g_i}$.

- $\boxed{\ker \partial_2 = \langle Sq^1 g_0, Sq^4 g_0 + Sq^2 g_1 + Sq^1 g_2 \rangle_{\mathcal{A}}}$ (optional comment)

Continuing on we get

$$0 \leftarrow \mathbb{F}_2 \leftarrow \mathcal{A}_{\leq 6} \leftarrow \mathcal{A}\{h_0, h_1, h_2\}_{\leq 6} \leftarrow \mathcal{A}\{g_0, g_1, g_2\}_{\leq 6} \leftarrow \mathcal{A}\{f_0, f_1\}_{\leq 6} \leftarrow \mathcal{A}\{e_0\}_{\leq 6} \leftarrow \mathcal{A}\{d_0\}_{\leq 6} \leftarrow \mathcal{A}\{c_0\}_{\leq 6} \leftarrow 0$$

where

$|h_0| = 1$, $|h_1| = 2$, $|h_2| = 4$

$|g_0| = 2$, $|g_1| = 4$, $|g_2| = 5$

$|f_0| = 3$, $|f_1| = 6$

$|e_0| = 4$

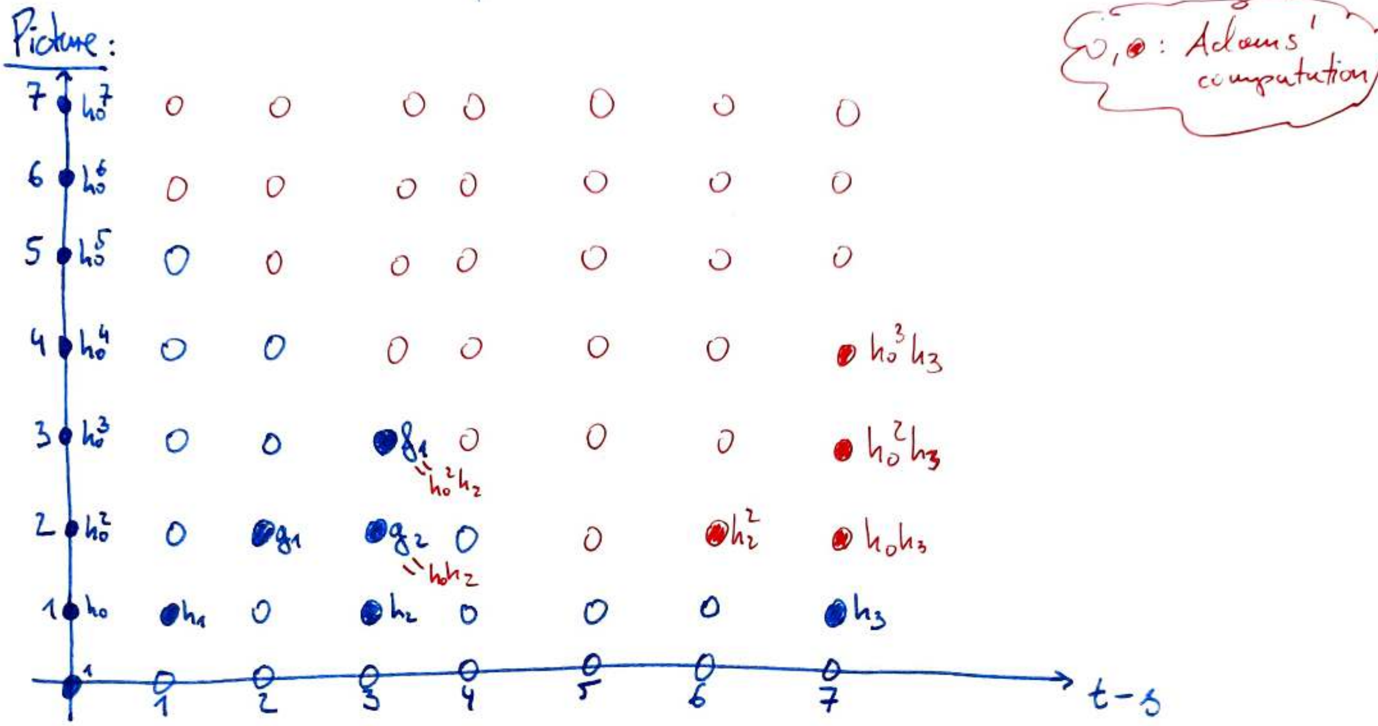
$|d_0| = 5$

$|c_0| = 6$

Therefore, in the range $0 \le s, t \le 6$, the only "new" nontrivial $\mathrm{Ext}_{\mathcal{A}}$ terms are

$$\mathrm{Ext}_{\mathcal{A}}^{2,4}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 g_1, \quad \mathrm{Ext}_{\mathcal{A}}^{3,5}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 f_1, \quad \mathrm{Ext}_{\mathcal{A}}^{2,5}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 g_2.$$

Picture:



Adams went further and computed the E_2 page for $0 \le s, t-s \le 7$ and used the multiplicative structure as well.

- We see that in every column everything eventually becomes 0 which is unsurprising as π_n^s is finite for $n \ge 1$. (filtration stabilizes $\Rightarrow$ Ext eventually 0)

Adams proved a stronger result, namely that everything above roughly the line with slope $\frac{1}{2}$ is 0.

THM: [Adams' vanishing line]

Let
$$f(s) := \begin{cases} 2s-1 ; & s \equiv 0,1 \mod 4 \\ 2s-2 ; & s \equiv 2 \mod 4 \\ 2s-3 ; & s \equiv 3 \mod 4 \end{cases}$$

Then
$$\boxed{0 < t-s < f(s) \Rightarrow \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0}.$$

Denote $\pi_n := (\pi_n^S)^{\wedge}_2$. Recall that for $n \geq 1$, π_n^S finite $\Rightarrow$ $\boxed{\pi_n \text{ is just the } 2\text{-torsion of } \pi_n^S}$.

We can conclude

$\pi_0 = \mathbb{Z}^{\wedge}_2$, $\pi_1 = \mathbb{Z}/2$, $\pi_2 = \mathbb{Z}/2$, $\pi_4 = 0$, $\pi_5 = 0$, $\pi_6 = \mathbb{Z}/2$.

For π_3 & π_7 we have to solve extension problems.

· $\pi_3 \cong \mathbb{Z}/8$: Filtration

$$\pi_3 = F^0\pi_3 \supseteq F^1\pi_3 \supseteq F^2\pi_3 \supseteq F^3\pi_3 \supseteq \dots$$

· $Ext^{0,3} = 0 \Rightarrow F^0\pi_3 = F^1\pi_3$

· $n \geq 4 \Rightarrow Ext^{n,n+3} = 0 \Rightarrow F^4\pi_3 = F^5\pi_3 = \dots = 0$.

Take $\nu \in \pi_3$. ν is detected by $h_2 \neq 0$ of $[\nu] \in F^1\pi_3 / F^2\pi_3$

$\Rightarrow \boxed{\nu \in F^1\pi_3 \ \& \ \notin F^2\pi_3}$

Similarly, $(\cdot 2) \in F^1\pi_0$ is detected by h_0.
By mult. of the filtr., we have that

$$(\cdot 2)\nu \in F^2\pi_3, \quad (\cdot 2)^2\nu \in F^3\pi_3, \quad \underset{\in F^4\pi_3 = 0}{(\cdot 2)^3\nu = 0}.$$

Diagram:

$$\begin{tikzcd}
(\cdot 2)\otimes\nu \arrow[rr, mapsto] \arrow[dd, bend right] & & (\cdot 2)\nu \arrow[dd, bend left] \\
F^1\pi_0/F^2\pi_0 \otimes F^1\pi_3/F^2\pi_3 \arrow[rr] \arrow[d, "\cong"'] & & F^2\pi_3/F^3\pi_3 \arrow[d, "\cong"] \\
E_\infty^{1,1} \otimes E_\infty^{1,4} \arrow[rr] & & E_\infty^{2,5} \\
h_0 \otimes h_2 \arrow[rr, mapsto] & & h_0h_2 \neq 0 \text{ (Adams)}
\end{tikzcd}$$

$\Rightarrow \boxed{(\cdot 2)\nu \neq 0}$ Similarly, $\boxed{(\cdot 2)^2\nu \neq 0}$.

Also, by unpacking mult. in π_n^S, we can see $(\cdot 2)^2\nu = 2\cdot(\cdot 2)\nu = (\cdot 2)\nu + (\cdot 2)\nu$.

· $0 \to \mathbb{F}_2 \cong F^3\pi_3 \longrightarrow F^2\pi_3 \longrightarrow F^2\pi_3/F^3\pi_3 \to 0$.

$F^3\pi_3 \cong E_\infty^{3,6} \cong E_2^{3,6} \cong \mathbb{F}_2$ $\qquad F^2\pi_3/F^3\pi_3 \cong E_\infty^{2,5} \cong \mathbb{F}_2$

This ext. is elt. of $Ext^1_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$, so it is either split or $F^2\pi_3 \cong \mathbb{Z}/4$.
Since $(\cdot 2)^2\nu \neq 0$, this elt $(\cdot 2)\nu$ is <u>not</u> 2-torsion. $\Rightarrow$ <u>not split</u>!

· $0 \to \mathbb{Z}/4 \longrightarrow F^1\pi_3 = \pi_3 \longrightarrow F^1\pi_3/F^2\pi_3 \cong \mathbb{F}_2 \to 0$.

$Ext^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/4) \cong \mathbb{Z}/2 \Rightarrow$ For same reason as above, not split. ν is not 4-tor.

$\Rightarrow \boxed{\pi_3 \cong \mathbb{Z}/8}$