Connections to topological K-theory

3.3.7. Topological K-theory of spaces

Def.: A complex vector bundle over a topological space X is a map $\pi : E \to X$ together with

① for all $x \in X$, a complex vector space structure on the fiber $E_x := \pi^{-1}(x) \cong \mathbb{C}^m$ (where m may depend on x!)

② a covering $\{U_i\}_{i\in I}$ of X by open subsets and homeomorphisms $\{h_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^{m_i}\}_{i\in I}$

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "h_i"] \arrow[d, "\pi|_{\pi^{-1}(U_i)}"'] & U_i \times \mathbb{C}^{m_i} \arrow[dl, "pr_1"] \\ U_i & \end{tikzcd}$$

commutes and $(h_i)_x : E_x \to \{x\} \times \mathbb{C}^{m_i}$ is a linear isomorphism $\forall x \in U_i$.

We usually write a vector bundle as a triple (E, π, X).

Note that E_x has the same dimension for all x in the same path component.

A map of vector bundles over X, $f : (E, \pi, X) \to (E', \pi', X)$ is a map $f : E \to E'$ such that

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

commutes and $f|_{E_x} : E_x \to E'_x$ is a linear map. $\forall x \in X$.

We get therefore a category of complex vector bundles over X, denoted $Vect(X)$.

Example: $\varepsilon_m = (X \times \mathbb{C}^m, pr_1, X)$ is called the trivial bundle over X.
A vector bundle is called trivial whenever it is isomorphic to the trivial bundle.
The tangent bundle of a manifold (TM, π, M).

The Whitney sum of $\xi = (E, \pi, X)$, $\eta = (F, g, X) \in \mathrm{Vect}(X)$ is the vector bundle $(G, \nu, X) \cong \xi \oplus \eta$, where

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\ F \arrow[r, "g"'] & X \end{tikzcd}$$

is a pullback diagram in Top, and ν is the composition $(G \to E \xrightarrow{\pi} X) = (G \to F \xrightarrow{g} X)$.

Note, that $G_x \cong E_x \oplus F_x$. There are other constructions of vector spaces we wish to similarly translate fiberwise to constructions on vector bundles.

The clutching construction (aka descent)

Let $\{U_i\}_{i \in I}$ be an open cover of $X \in \mathrm{Top}$. Let $\xi_i = (E_i, \pi_i, U_i)$ be a vector bundle over each U_i and let $g_{ji} : \xi_i|_{U_i \cap U_j} \to \xi_j|_{U_i \cap U_j}$ be isomorphisms such that they satisfy the cocycle condition:

$$\begin{tikzcd} \xi_k|_{U_i \cap U_j \cap U_k} \arrow[rr, "g_{ki}|_{U_i \cap U_j \cap U_k}"] \arrow[dr, "g_{kj}|_{U_i \cap U_j \cap U_k}"'] & \circlearrowleft & \xi_i|_{U_i \cap U_j \cap U_k} \\ & \xi_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "g_{ji}|_{U_i \cap U_j \cap U_k}"'] & \end{tikzcd}$$

This in particular means, that $s \times t : \underbrace{\coprod_{(i,j) \in I \times I} \xi_i|_{U_i \cap U_j}}_{R} \to \underbrace{\left(\coprod_{i \in I} \xi_i\right)}_{\tilde{\xi}} \times \underbrace{\left(\coprod_{j \in I} \xi_j\right)}_{\tilde{\xi}}$ gives an equivalence relation:

$\forall e, e' \in \tilde{E}$ $(e \sim e' \Leftrightarrow \exists r \in R$ s.t. $s(r) = e \wedge t(r) = e')$. $(\tilde{\xi} = (\tilde{E}, \tilde{\pi}, X))$

Here we set $s = \coprod_{(i,j) \in I \times I} (\xi_i|_{U_i \cap U_j} \hookrightarrow \xi_i \hookrightarrow E)$ and

$$t = \coprod_{(i,j) \in I \times I} (\xi_i|_{U_i \cap U_j} \xrightarrow{g_{ij}} \xi_j|_{U_i \cap U_j} \hookrightarrow \xi_j \hookrightarrow E).$$

Now we construct the bundle $\xi = (E, \pi, X)$ with $E = \left(\coprod_{i \in I} E_i\right) / (e_i \sim e_j \Leftrightarrow g_{ij}(e_i) = e_j)$
$= \tilde{\xi}/\sim$

ξ is up to isomorphism the unique vector bundle s.t. there are isomorphisms $g_i : \xi_i \to \xi|_{U_i}$ making the diagrams

$$\begin{tikzcd} \xi_i|_{U_i \cap U_j} \arrow[rr, "g_{ji}"] \arrow[dr, "g_i|_{U_i\cap U_j}"'] & & \xi_j|_{U_i \cap U_j} \arrow[dl, "g_j|_{U_i\cap U_j}"] \\ & \xi|_{U_i \cap U_j} & \end{tikzcd}$$

commute.

Using the clutching construction we can consider two vector bundles $\xi = (E, \pi, X)$ and $\eta = (F, g, X)$ so that they are both trivialized over a covering $\{U_i \subseteq_{\text{open}} X\}_{i \in I}$. Then we can form the fiber-wise

- the (fiberwise) tensor product $\xi \otimes \eta$ is such that $(\xi \otimes \eta)|_{U_i} = (\xi|_{U_i}) \otimes (\eta|_{U_i})$ where we use that one can easily form the tensor product of trivial bundles: $(X \times \mathbb{C}^m, pr_1, X) \otimes (X \times \mathbb{C}^n, pr_1, X) = (X \times \mathbb{C}^m \otimes \mathbb{C}^n, pr_1, X)$.
- hom space $Hom(\xi, \eta)$ with $Hom(\xi,\eta)|_{U_i} = Hom_{\mathbb{C}}(\xi|_{U_i}, \eta|_{U_i})$
- etc.

Constructing $K^0_{\mathbb{C}}(X)$

Crucially, $(\underbrace{Vect(X)/\cong}_{\text{isomorphism classes of vector bundles}}, \oplus)$ is an abelian monoid and $\otimes$ distributes over $\oplus$.

One can apply to $(Vect(X)/\cong, \oplus)$ the Grothendieck construction and get the group $K^0_{\mathbb{C}}(X) \in Ab$.

The tensor product provides a multiplication on $K^0_{\mathbb{C}}(X)$ making it a (unital) commutative ring.

If $f : Y \to X$ is a map of topological spaces, we get a functor

$$f^* : Vect(X) \to Vect(Y)$$
$$(E, \pi, X) \longmapsto (E', \pi', Y)$$

where

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

is a pullback diagram.

This induces a ring homomorphism $f^*: K^0_{\mathbb{C}}(X) \to K^0_{\mathbb{C}}(Y)$.

Stable equivalence of vector bundles:
$\xi, \eta \in Vect(X)$ are equivalent if there is a trivial bundle $\varepsilon \in Vect(X)$ such that $\xi \oplus \varepsilon = \eta \oplus \varepsilon$.

<u>Theorem</u>: If X is compact, then for any $\xi \in Vect(X)$ there is some $\eta \in Vect(X)$ such that $\xi \oplus \eta$ is a trivial bundle.

Group completion is left adjoint to the forgetful functor $Ab \to CMon$
So there is a cannonical map of monoids

$$\begin{tikzcd} (Vect(X)/\cong, \oplus) \arrow[d, "s"'] \arrow[dr, "\alpha"] & \\ K^0_{\mathbb{C}}(X) \arrow[r] & G \end{tikzcd}$$

such that any group G and map of monoids $\alpha: (Vect(X)/\cong, \oplus) \to G$ there is a unique map of groups making the diagram of monoids commute.

Let us denote $\mathcal{E}(X) = (Vect(X)/\cong, \oplus)$ if we consider some $\xi, \xi' \in \mathcal{E}(X)$ such that there is some $\eta \in \mathcal{E}(X)$ with $\xi \oplus \eta = \xi' \oplus \eta$ this means that $s(\xi) + s(\eta) = s(\xi') + s(\eta)$ so $s(\xi) = s(\xi')$.
In fact unpacking Grothendieck's construction this is an if and only if.

We get familiar standard pictures:

$$K^0_{\mathbb{C}}(X) = \{ [\xi] - [\eta] \mid \xi, \eta \in Vect(X)/\cong \}$$

if X is compact, then $K^0_{\mathbb{C}}(X) = \{ [\xi] - [\varepsilon_n] \mid \xi, \varepsilon_n \in Vect(X)/\cong \text{ and } \varepsilon_n \text{ is trivial} \}$.

<u>Example</u>: Consider $X = *$, then $K^0_{\mathbb{C}}(*) \cong \mathbb{Z} = \{ [\varepsilon_n] - [\varepsilon_m] \mid \varepsilon_n \in Vect(*)/\cong \}$

$$n - m \longleftrightarrow [\varepsilon_n] - [\varepsilon_m]$$

The only vector bundles over X are the trivial ones.

K-theory and Gelfand duality

locally compact Hausdorff

$$C_0 : \mathrm{LCHaus}^{op} \xrightarrow{\simeq} C^*\text{-Alg}_{com}$$
$$X \longmapsto C_0(X)$$
$$\{X \xrightarrow{f} \mathbb{C} \text{ s.t. } \forall \varepsilon>0\ \exists K \underset{cpt.}{\subseteq} X \text{ s.t. } \forall x \in X \setminus K\ |f(x)| < \varepsilon\}$$

is an equivalence of categories via the Gelfand duality

X is compact $\Leftrightarrow C_0(X)$ is unital

Consider X compact now $C_0(X) = C(X) = \{f : X \to \mathbb{C} \text{ continuous}\}$.
We have a map

$$P_\infty(C(X)) \longrightarrow \mathrm{Vect}(X)/_{\cong}$$
$$p \longmapsto \xi_p = (E_p, \pi, X)$$

as follows: suppose $p \in P_n(C(X))$, since $M_n(C(X)) \cong C(X, M_n(\mathbb{C}))$ we can take

$$E_p = \{(x,v) \in X \times \mathbb{C}^n \mid \exists w \in \mathbb{C}^n \text{ s.t. } p(x)(w) = v\}$$

$\pi : E_p \to X$ shall be the projection

$$(E_p)_x = p(x)(\mathbb{C}^n) \subseteq \mathbb{C}^n$$

__Claim:__ $\forall p,q \in P_\infty(C(X))$

① $\xi_p \cong \xi_q \Leftrightarrow p \sim_0 q$

② $\xi_{p \oplus q} \cong \xi_p \oplus \xi_q$

__Proof:__

① ($\Leftarrow$) If there is some $v \in M_{m,n}(\mathbb{C})$ s.t. $p = vv^*$ and $q = v^*v$, then we get

$$E_p \longrightarrow E_q$$
$$(x, p(x)(w)) \longmapsto (x, v^*(x)(p(x)(w)))$$

as $v^*(x)(v(x)v^*(x))(w)$
$= \underbrace{(v^*(x)v(x))}_{q(x)} v^*(x)(w)$

which admits the inverse

$$E_q \longrightarrow E_p$$
$$(x, q(x)(w)) \longmapsto (x, v(x)(q(x)(w)))$$

($\Rightarrow$) ?

② $\xi_{p\oplus q} = (E_{p\oplus q}, \pi, X)$

$$E_{p\oplus q} = \{ (x,v) \in X \times \mathbb{C}^{n+m} \mid \exists w \in \mathbb{C}^{n+m} \text{ s.t. } \underbrace{p\oplus q}_{\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}(w) = v \}$$

so really $(E_{p\oplus q})_x = (E_p)_x \oplus (E_q)_x$.

$\square$

This means that we get a well-defined and injective map of monoids

$$\begin{aligned} D(C(X)) &\longrightarrow (\mathrm{Vect}(X)/\cong, \oplus). \\ [p]_0 &\longmapsto \xi_p \end{aligned}$$

Furthermore, if I have a vector bundle $\xi = (E, \pi, X)$ and some $\eta = (F, v, X)$ such that $\xi \oplus \eta = \varepsilon_n$, then in fact I get on each fiber a projection

$$p(x): E_x \oplus F_x \cong \mathbb{C}^n \longrightarrow E_x$$

This provides some $p \in C(X, M_n(\mathbb{C}))$ that is in $\mathcal{P}_n(C(X))$ so that $\xi \cong \xi_p$.

Hence this map is also surjective and thus $D(C(X)) \cong (\mathrm{Vect}(X)/\cong, \oplus)$. Group completion is functorial and therefore $K^0_{\mathbb{C}}(X) \cong K_0(C(X))$.

<u>Remark</u>: Recall that we have a ring structure on $K^0_{\mathbb{C}}(X)$. This does not generalize to a ring structure on K-groups of arbitrary C^*-algebras.

11.4. Homotopy groups and K-theory

$uu^* = u^*u = 1$ unitary $n\times n$ matrices with entries from A

Let $\mathcal{U}_\infty(A) = \underset{n}{\mathrm{colim}}(\mathcal{U}_n(A))$ where A is a unital C^*-algebra and the map $\mathcal{U}_n(A) \to \mathcal{U}_{n+1}(A)$ is $u \longmapsto u \oplus 1_1$. ← these are isometries!

$\mathcal{U}_\infty(A)$ is a topological group: $d(u,v) = \|u \oplus 1_n - v \oplus 1_m\|$ if $u \in \mathcal{U}_m(A)$ and $v \in \mathcal{U}_n(A)$.

Consider $\mathcal{V}(A) = \{ u \in \mathcal{U}_\infty(A) \mid s(u) = 1 \}$ where $s: \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(\mathbb{C})$

is the scalar mapping. If A is unital, then $\mathcal{V}(A) \cong \mathcal{U}_\infty(A)$.

Given a topological group G, $\pi_k(G) = \{S^k \to G\}/\simeq$ relative to basepoints endowed with pointwise multiplication.

<u>Proposition 11.4.1.</u> Let A be a C^*-algebra. $\forall n \geq 0$ $K_n(A) \cong \pi_{n-1}(\mathcal{V}(A))$.

<u>Sketch proof:</u>

one has to work for being able to quotient by $\sim_h$ instead of $\sim_1$

$$\pi_0(\mathcal{V}(A)) = \mathcal{V}(A)/\sim_h \cong \mathcal{U}_\infty(\tilde{A})/\sim_h \cong K_1(A)$$

$S^n \cong [0,1]^n/\partial[0,1]^n$, let $T_n^1(A) = \{f \in C([0,1]^n, \mathcal{V}(A)) \mid f(t) = 1 \ \forall t \in \partial[0,1]^n\}$.

$\pi_n(\mathcal{V}(A)) \cong T_n^1(A)/\sim_h$ and $T_1^1(A) \cong \mathcal{V}(SA)$.

$$\pi_1(\mathcal{V}(A)) \cong T_1^1(A)/\sim_h = \mathcal{V}(SA)/\sim_h \cong K_1(SA) \cong K_2(A). \quad \square$$

$\mathcal{U} = \operatorname{colim}_n \mathcal{U}_n(\mathbb{C})$ this provides $\pi_{n+2}(\mathcal{U}) \cong \pi_n(\mathcal{U})$ for $A = \mathbb{C}$.

Let X be compact, then

$SC(X) \cong C(SX)$ so unital

$SX = (X \times [0,1])/(X \times \{0\} \amalg X \times \{1\})$

$$K^0_{\mathbb{C}}(X) \cong K_0(C(X)) \cong K_2(C(X)) \cong \mathcal{V}(SC(X))/\sim_h \cong \mathcal{U}_\infty(C(SX))/\sim_h$$
$$\cong [SX, \mathcal{U}] \cong [X, \Omega\mathcal{U}]$$

$\Omega\mathcal{U} = C(S^1, \mathcal{U})$

We ignore basepoints here so we should assume, that X is a connected CW-complex.

Bott periodicity stated in a topological way:

Now I am using reduced suspension and loops and pointed homotopy classes of maps!

$$\Omega\mathcal{U} \simeq \mathbb{Z} \times B\mathcal{U} \quad \text{where} \quad \Omega B\mathcal{U} \simeq \mathcal{U}$$

$$K^n_{\mathbb{C}}(X) \cong [\Sigma^n X, \Omega\mathcal{U}]_* \cong [X, \Omega^{n+1}\mathcal{U}]_* \cong \begin{cases} [X, \mathcal{U}]_* & \text{if } n \text{ odd} \\ [X, \mathbb{Z} \times B\mathcal{U}]_* & \text{if } n \text{ even} \end{cases}$$

<u>Remark on algebraic K-theory</u>

<u>Theorem (Serre-Swan):</u> If X is a compact Hausdorff space, then there is an equivalence of categories

$$\begin{tikzcd} \mathrm{Vect}(X) \arrow[r, "\widetilde{\Gamma}(X,-)"] \arrow[d, "\Gamma(X,-)"'] & \mathrm{Vect}(\mathrm{Spec}(C(X,\mathbb{C}))). \\ C(X,\mathbb{C})\text{-Mod} \arrow[ur, "\widetilde{(-)}"'] & \end{tikzcd}$$

quasi coherent locally free of finite rank $\mathcal{O}_{C(X,\mathbb{C})}$-modules