Plan for today:

→ finish Chapter 10

- Recap Gibbons Hawking
- Belinski-Gibbons-Page-Pope
- Other instantons
  - Atiyah Hitchin
  - Compact + K3.

→ Dunajski 2025.

- Toric instantons → KL 2025.
  - rod structures
  - ASDYM ⇝ Yang Eqn
  - Twistor Construction

Gibbons Hawking:

M = total space of S^1 bundle over $\mathbb{R}^3 - X$
(extends to $\mathbb{R}^3$)

- $V g_{\mathbb{R}^3} + \eta \leftarrow$ Connection 1-form on S^1 bundle.

- Self dual 2-forms
  $$\Omega^i = dx^i \wedge \eta + V *_3 (dx^j \wedge dx^k)$$

- $d\Omega^i = 0 \Rightarrow d\eta = *_3 dV$
  Set $\eta = d\tau + A$
  τ = fibre coordinate

- $$V = V_0 + \sum^{M} \frac{1}{|x - x_m|}$$

<u>Thm</u>: Tri-holomorphic Killing field
$\Rightarrow$ Gibbons hawking. (local)

<u>Proof</u>:

$$g = h + V^{-1}(d\tau + A)$$

$\uparrow$ killing field orbits.

Where $V = g(K,K)^{-1}$

$$0 = L_K \Omega_i \Rightarrow \cancel{K \lrcorner d\Omega_i} + d(K \lrcorner \Omega_i) = 0$$
$$\Rightarrow d(K \lrcorner \Omega_i) = 0$$
$$\Rightarrow K \lrcorner \Omega^i = dx^i$$

these are <u>Hamiltonians</u> for action of K.

(tool from symplectic geom, gives conserved quantities).

Hermitian $g(I_i(K), I_i(K)) = g(K,K)$
but LHS $= |dx^i|^2$

and $\quad g_{ab} k^a k^b = g^{ab} \nabla_a x^1 \nabla_b x^1$

$$= g^{ab} \nabla_a x^2 \nabla_b x^2$$

$$= g^{ab} \nabla_a x^3 \nabla_b x^3$$

$\nmid \quad g^{ab} \nabla_a x^1 \nabla_b x^2 = 0$

$$\Rightarrow \quad g = V^{-1} (d\tau + A) + V \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right)$$

$g^* I_i = I_i$

BGPP Class, triholom SU(2) action.

on $M = \mathbb{R} \times SU(2)$ all hyperkähler metrics

$$g = w_1 w_2 w_3 \, dp^2 + \sum_{i=1}^{3} \frac{w_1 w_2 w_3}{w_i^2} (\sigma_i)^2$$

$$\Omega_1 = w_1 \sigma_2 \wedge \sigma_3 + w_2 w_3 \sigma_1 \wedge dp$$

ETC.

Assume: SU(2) action fixes complex structures

$$\Rightarrow \quad d\Omega_i = 0$$

This gives Euler eqns (Admits Lax pair!)

$$\dot{w}_1 = w_2 w_3, \quad \dot{w}_2 = w_1 w_3, \quad \dot{w}_3 = w_1 w_2.$$

N.B.: $\frac{d}{d\tau}\left(w_i^2 - w_j^2\right) = \dot{w}_i w_i - \dot{w}_j w_j = 0$

$$\Rightarrow (\dot{w}_3)^2 = w_1^2 w_2^2 = (w_3^2 - c_1)(w_3^2 - c_2)$$

$$= w_1^2 - w_3^2$$

this metric is flat if $c_1 = 0 = c_2$.

Not complete $c_1 c_2 (c_1 - c_2) \neq 0$

turns out other cases $\Rightarrow$ Eguchi-Hanson.

When does Gibbons Hawking $(V, A) \subseteq$ BGPP?

Pick $U(1) \subseteq SU(2)$. this is $\therefore$ Gibbons Hawking

Write in this form, $K = K^a \sigma_a$ and then

$V = g(K, K)^{-1}$ implies

$$V = \prod^3 (\beta - \beta_i)^{-1/2} \left(\sum \frac{x_i^2}{(\beta - \beta_i)^2} \right)^{-1}$$

β satisfies $\sum x_i^2 / (\beta - \beta_i)^2 = C$

<u>Equivalently</u>

$V = r \cdot \hat{V}$ where $B_{ij}(\hat{v}) x^i x^j = \text{Const}.$

Ex: Eguchi Hanson

$$V = |r+a|^{-1} + |r-a|^{-1}$$

$$\hat{V} = -\frac{2}{a} \operatorname{arccoth}\left(\frac{|r+a| + |r-a|}{2a} \right)$$

Other Instantons (No AE, ALE?)

Positive action Thm: Given AE instanton, if $R \geq 0$ everywhere then

$I \geq 0$ equality iff flat.

ALE: $\Gamma \leq SU(2)$ finite

these correspond to platonic solids.
for each Γ $\exists$ invariants x, y, z polyn in z_1, z_2 on $\mathbb{C}^2$, then $\mathbb{C}^2/\Gamma \leq \mathbb{C}^3 = \{(x,y,z) : F_\Gamma(x,y,z) = 0\}$

Group	F_Γ.
Cyclic,	$xy - z^k = 0$
dihedral,	$x^2 + y^2 z + z^k = 0$
tetrahedral,	$\vdots$
octahedral,	
Icosahedral	

Manifold M is minimal resolution of $\mathbb{C}^2/\Gamma$.

i.e. Zero set of $\tilde{F}_\Gamma = F_\Gamma + \sum a_i f_i$

f_i span ring of polynomials nonvanishing at $\partial_x F_\Gamma = \partial_y F_\Gamma = \partial_z F_\Gamma = 0$.

e.g. $\mathbb{Z}_2 = \left\langle \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right\rangle \subseteq SU(2)$.

then $x = (z_1)^2$, $y = z_2^2$ $z = z_1 z_2$.

$M = Z(xy - z^2)$

$dF = (y, x, -2z)$ Vanish at $(0,0,0)$

perturb $xy - (z-p_1)(z-p_2) = \tilde{F}_\Gamma$

then $Z(\tilde{F}_\Gamma)$ is Eguchi-Hanson mfld.

<u>Thm</u>: Kronheimer: $\exists!$ hyperkähler metric on $\mathbb{C}^2/\Gamma$ for each $\Gamma \leq SU(2)$ finite

ALF example (Taub NUT)

Another one is Atiyah Hitchin

Assume instead $SU(2)$ rotates Ω_i, then

$$d\Omega_i = \alpha_{ij} \wedge \Omega_j \quad \text{then}$$

$$R_{A'}{}^{B'} = 0 \Rightarrow \dot{w}_1 = w_2 w_3 - w_1(w_2 + w_3)$$

ETC.

this is solved using elliptic functions

$$g = \frac{w_1 w_2 w_3}{\bar{W}^4} dp^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \cdots$$

$$w_1 = -W \frac{dW}{dp} - \frac{1}{2} W^2 \operatorname{cosec} p$$

$$w_1 = -W \frac{dW}{dp} + \frac{1}{2} W^2 \cot p$$

$$w_1 = -W \frac{dW}{dp} + \frac{1}{2} W^2 \operatorname{cosec} p$$

$$\frac{d^2W}{dp^2} + \frac{1}{4} W \operatorname{cosec}^2 p = 0$$

$$W = \frac{1}{\pi} \sqrt{\sin p \, K(\sin^2 p/2)}$$

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k \sin^2 \phi}}$$

→ Natural metric on moduli space of charge-two nonabelian magnetic SU(2) monopoles.

geodesics = low energy monopole scatt.

Compact Instantons

Compact - Einstein is easy

- Fubini study metric on $\mathbb{C}P^n$

- Conformally flat S^4

$$g = \frac{1}{(1+(\rho/r_0)^2)^2}\left(d\rho^2 + \frac{\rho^2}{4}(\sigma_1^2+\sigma_2^2+\sigma_3^2)\right)$$

WB Ricci flat?

$\rightarrow \quad T^m = \underbrace{S^1 \times \cdots \times S^1}_{m\text{-times}}.$

<u>Problem</u>: $\nexists$ KVF on Ricciflat compact except for on S^1 factors.

$$\nabla_a \nabla_b K_c = R_{bca}{}^d K_d$$

$$\Rightarrow \quad R_{ab} k^a k^b - k^b \Box k_b = 0$$

$$\Rightarrow \quad \int_M \left(R_{ab} k^a k^b + |\nabla_a k_b|^2 \right) dvol_g. = 0$$

$$\Rightarrow \quad \nabla_a k_b = 0 \quad \text{on } M$$

then in coords adapted to $k = \partial_\tau$

then $g = d\tau^2 + g_3$ $\quad g_3$ Ricci flat $\Rightarrow$ flat. $\quad \Box$.

K3 surface known to exist (by Yau's proof of Calabi Conjecture).

$\forall\, \rho_1 \in 2\pi c_1(M)\ \exists!$ Kahler metric with $\omega_1 \in [\omega]$ ricci form ρ_1

Toric instantons

$k_i = \partial_{\phi^i}$ Commuting Killing fields

$\phi_i \to 2\pi$ periodic.

Weyl-Papapetrou coords

$$g = \Omega^2 (d\rho^2 + dz^2) + G_{ij}\, d\phi^i d\phi^j$$

$$\rho^2 = \det G \quad , \quad *_2 dz = d\rho .$$

"Harmonic coords"

Orbit space M/T ← (torus action)

is 2-d Simply connected manifold with bdd & corners.

global coords $\{(\rho, z) : \rho \geq 0\} = \mathbb{H}$.

$\partial\mathbb{H}$ is $\rho = 0$, then

$$\text{rank } G = \begin{cases} 1 & \text{Edges} \\ 0 & \text{Corners} \end{cases}$$

Note: finitely many Corners, (Isolated).

$I_1 = (-\infty, z_1)$, $I_2 = (z_1, z_2), \ldots, I_{N+1} = (z_N, \infty)$

rod structure:

- $\ell_A = z_{A-1} - z_A \qquad A = 2, \ldots$
- v_A = vector vanishing on rod.
  ($\mathbb{Z}$-linear Combo of ∂_{ϕ^i})

Admissible $\det \begin{pmatrix} v_k^1 & v_k^2 \\ v_{k+1}^1 & v_{k+1}^2 \end{pmatrix} = \pm 1$

Rod structure fixes topology

[Kunduri-Lucietti $\exists$ at most 1 AF instanton per rod structure.]

Ricci flatness

$$\Rightarrow \begin{cases} d\hat{*}(\rho J) = 0 \\ \hat{R}_{ab} = \hat{\nabla}_a \hat{\nabla}_b \log\rho + \frac{1}{4}\operatorname{tr} J_a J_b = 0 \end{cases}$$

$$J = G^{-1} dG$$

$$dJ + J \wedge J = 0$$

In coords

$$\text{Yang} \leftarrow r^{-1}\partial_r(rJ^{-1}\partial_r J) + \partial_z(J^{-1}\partial_z J) = 0$$

Ω from one integration.

in ASDYM $\quad g = 2(dz\,d\tilde{z} - dw\,d\tilde{w})$

$$F = d\Phi + \Phi \wedge \Phi$$

ASDYM are $\quad F_{wz} = 0 \quad F_{\tilde{w}\tilde{z}} = 0 \quad F_{w\tilde{w}} = F_{z\tilde{z}}$

Gauge

first two $\Rightarrow$ $\Phi = J^{-1}\partial_{\tilde{w}}(J d\tilde{w}) + J^{-1}\partial_{\tilde{z}} J d\tilde{z}$

$$z = t + z \qquad w = re^{i\theta}$$
$$\tilde{z} = t - z \qquad \tilde{w} = re^{-i\theta}$$

& symmetry reduction $J = J(r, z)$ reduces to Yang eqn.

then use WARD CORR.