Large-spin perturbation theory and kicking off the thermal bootstrap

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1 Recap of thermal bootstrap formulae

- So far, we have seen two formulae for the thermal bootstrap. The inversion formula:

$$a(\Delta, J) = \frac{(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})} \int_0^\beta \mathrm{d}\bar{z} \int_\beta^{\beta^2/\bar{z}} \mathrm{d}z (z\bar{z})^{\Delta_\phi - \frac{\Delta+d}{2}} (z-\bar{z})^{d-2} F_J\left(\sqrt{\bar{z}/z}\right) \times \beta^\Delta \mathrm{Disc}[g(z,\bar{z})] \tag{1}$$

  where the function F_J is defined as

$$F_J(x) = x^{J+d-2} {}_2F_1\left(J+d-2, \frac{d}{2}-1, J+\frac{d}{2}, x^2\right) \tag{2}$$

- This expression tells you the thermal data in terms of the analytic structure of the two-point function; but you need every single pole in $a(\Delta, J)$ to reconstruct g.

- As an aside, it can also clarify the notion of the 'out-of-OPE' contribution to the one-point functions. Due to KMS, we have $g(z,\bar{z}) = g(\beta - z, \beta - \bar{z})$; the OPE of the latter converges only if $|z-\beta| = |\bar{z}-\beta| \leq \beta$, but the inversion formula also requires contributions from $z > 2\beta$. These are called 'out-of-OPE' contributions.

- Last week, Ilias introduced an alternative perspective through the dispersion relation. The aim of this relation is to express g in terms of its own analytic structure directly:

$$g(\tau, \mathbf{0}) = \sum_{n\in\mathbb{Z}} \int_{-i\infty}^{0} \frac{\mathrm{d}\tau'}{2\pi i} \frac{\mathrm{Disc}[g(\tau', \mathbf{0})]}{\tau' + n\beta - \tau} + g_{\mathrm{arcs}}(\tau, \mathbf{0}) \tag{3}$$

  This comes from Cauchy's theorem followed by deforming the contour to wrap around the branch cuts starting at $\tau' = n\beta$ (due to KMS) and extending to infinity parallel to the negative complex axis (due to periodicity and invariance under $\tau \to -\tau$ of the correlator).

- Using the OPE expansion for a two-point function, you can explicitly find the discontinuity and do the integral and sum over n, resulting in

$$g(\tau, \mathbf{0}) = \sum_\Delta \frac{a_\Delta}{\beta^{2\Delta_\phi}} \left[\zeta_H\left(2\Delta_\phi - \Delta, 1 - \frac{\tau}{\beta}\right) + \zeta_H\left(2\Delta_\phi - \Delta, \frac{\tau}{\beta}\right)\right] + \kappa \tag{4}$$

  The sum of Hurwitz ζ-functions equals the (finite-temperature) propagator of an MFT operator with scaling dimension $2\Delta_\phi - \Delta$.

- A similar expression exists for $\mathbf{x} \neq \mathbf{0}$. This is useful because you can reconstruct the two-point function from its analytic structure; but, it doesn't tell you about thermal data.

- This week, I want to go over the actual bootstrap algorithm, using large-spin perturbation theory as an example.

2 Relations between one-point functions

- Recall: to derive the inversion formula, we started with the OPE decomposition of the two-point function as

$$\langle\phi(\tau,\mathbf{x})\phi(0)\rangle_\beta = g(\tau,\mathbf{x}) = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \frac{C_J^{(\frac{d-2}{2})}(\tau/|x|)}{|x|^{2\Delta_\phi-\Delta_{\mathcal{O}}}} \tag{5}$$

for $|x|^2 = \tau^2 + \mathbf{x}^2$. Using the complex coordinates $z = \tau + i|\mathbf{x}|$ and $\bar{z} = \tau - i|\mathbf{x}|$, this can be written as the s-channel representation of the two-point function:

$$g(z,\bar{z}) = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} (z\bar{z})^{\frac{\Delta_{\mathcal{O}}}{2}-\Delta_\phi} C_J^{(\frac{d-2}{2})}\left(\frac{1}{2}\sqrt{\frac{z}{\bar{z}}} - \frac{1}{2}\sqrt{\frac{\bar{z}}{z}}\right) \tag{6}$$

- The KMS condition now demands that $g(z,\bar{z}) = g(\beta - z, \beta - \bar{z})$, which means that we should also have the t-channel representation of the two-point function

$$g(z,\bar{z}) = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \left[(\beta-z)(\beta-\bar{z})\right]^{\frac{\Delta_{\mathcal{O}}}{2}-\Delta_\phi} C_{\ell_{\mathcal{O}}}^{(\frac{d-2}{2})}\left(\frac{1}{2}\sqrt{\frac{\beta-z}{\beta-\bar{z}}} - \frac{1}{2}\sqrt{\frac{\beta-\bar{z}}{\beta-z}}\right) \tag{7}$$

where we denoted the spin of operator $\mathcal{O}$ by $\ell_{\mathcal{O}}$ instead of J to prevent confusion later on.

- You can now expand the Gegenbauer function to get only a polynomial:

$$g(z,\bar{z}) = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \sum_{k=0}^{\ell_{\mathcal{O}}} p_k(\ell_{\mathcal{O}})(\beta-z)^{h_{\mathcal{O}}-\Delta_\phi+k}(\beta-\bar{z})^{\bar{h}_{\mathcal{O}}-\Delta_\phi-k} \tag{8}$$

where the quantum numbers $h_{\mathcal{O}}$ and $\bar{h}_{\mathcal{O}}$ are defined as

$$h_{\mathcal{O}} = \frac{\Delta_{\mathcal{O}} - \ell_{\mathcal{O}}}{2} = \frac{\tau_{\mathcal{O}}}{2} \qquad \bar{h}_{\mathcal{O}} = \frac{\Delta_{\mathcal{O}} + \ell_{\mathcal{O}}}{2} \tag{9}$$

and the coefficient $p_k(\ell_{\mathcal{O}})$ is, for completeness, given as

$$p_k(\ell_{\mathcal{O}}) = \frac{\Gamma(\ell_{\mathcal{O}} - k + \frac{d-2}{2})\Gamma(k + \frac{d-2}{2})}{\Gamma(\ell_{\mathcal{O}} - k + 1)\Gamma(k+1)} \frac{1}{\Gamma^2(\frac{d-2}{2})} \tag{10}$$

- On the other hand, you can write down a Taylor series for $(z-\bar{z})^{d-2}F_J(\sqrt{\bar{z}/z})$ and rewrite the inversion formula as

$$\begin{aligned} a(\Delta,J) = \frac{(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})} &\int_0^\beta \frac{\mathrm{d}\bar{z}}{\bar{z}} \int_\beta^{\beta^2/\bar{z}} \frac{\mathrm{d}z}{z} \sum_{m=0}^{\infty} q_m(J) z^{\Delta_\phi - \bar{h} - m} \bar{z}^{\Delta_\phi - h + m} \\ &\times \beta^\Delta \mathrm{Disc}[g(z,\bar{z})] \end{aligned} \tag{11}$$

where the coefficient (again for completeness) is defined as

$$q_m(J) = \frac{(-1)^m(J+2m)}{J} \frac{(J)_m(\frac{d}{2}-m)_m}{\Gamma(m+1)(J+\frac{d}{2})_m} \tag{12}$$

and, following the definition of $h_{\mathcal{O}}$ and $\bar{h}_{\mathcal{O}}$, we have set

$$h = \frac{\Delta - J}{2} \qquad \bar{h} = \frac{\Delta + J}{2} \tag{13}$$

- The idea is now to put the t-channel representation of g into the inversion formula (which was derived from the s-channel representation!) to get constraints from the equivalence of the two channels (this equivalence is the statement of the KMS condition). We get that

$$a(\Delta, J) = \sum_{\mathcal{O}} a^{(\mathcal{O})}(\Delta, J) \tag{14}$$

with the contributions $a^{(\mathcal{O})}(\Delta, J)$ given by

$$\begin{aligned} a^{(\mathcal{O})}(\Delta, J) &\approx \frac{(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})} \sum_{m=0}^{\infty}\sum_{k=0}^{\ell_{\mathcal{O}}} q_m(J) p_k(\ell_{\mathcal{O}}) \int_0^\beta \frac{\mathrm{d}\bar{z}}{\bar{z}} \int_\beta^{z_{\max}} \frac{\mathrm{d}z}{z}\, z^{\Delta_\phi - \bar{h} - m} \\ &\quad \times \beta^\Delta \bar{z}^{\Delta_\phi - h + m} (\beta - \bar{z})^{\bar{h}_{\mathcal{O}} - \Delta_\phi - k} \mathrm{Disc}[a_{\mathcal{O}}(\beta - z)^{h_{\mathcal{O}} - \Delta_\phi + k}] \beta^{-\Delta_{\mathcal{O}}} \\ &= \frac{(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})} \sum_{m=0}^{\infty}\sum_{k=0}^{\ell_{\mathcal{O}}} q_m(J) p_k(\ell_{\mathcal{O}}) \int_0^1 \frac{\mathrm{d}\bar{z}}{\bar{z}}\, \bar{z}^{\Delta_\phi - h + m} (1-\bar{z})^{\bar{h}_{\mathcal{O}} - \Delta_\phi - k} \\ &\quad \times 2 a_{\mathcal{O}} \sin(-\pi(h_{\mathcal{O}} - \Delta_\phi + k)) \int_1^{z_{\max}} \frac{\mathrm{d}z}{z}\, z^{\Delta_\phi - \bar{h} - m} (z-1)^{h_{\mathcal{O}} - \Delta_\phi + k} \\ &= \frac{a_{\mathcal{O}}(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})} \sum_{m=0}^{\infty}\sum_{k=0}^{\ell_{\mathcal{O}}} \frac{\Gamma(\Delta_\phi - h + m)\Gamma(\bar{h}_{\mathcal{O}} - \Delta_\phi - k + 1)}{\Gamma(\bar{h}_{\mathcal{O}} - h + m - k + 1)} \\ &\quad \times 2\pi q_m(J) p_k(\ell_{\mathcal{O}}) S_{h_{\mathcal{O}} - \Delta_\phi + k, \Delta_\phi - m}(\bar{h}) \end{aligned} \tag{15}$$

where in the second line we rescaled $z \to \beta z$ and $\bar{z} \to \beta \bar{z}$ and in the third line we defined

$$\begin{aligned} S_{c,\Delta}(\bar{h}) &= \frac{\sin(-\pi c)}{\pi} \int_1^{z_{\max}} \frac{\mathrm{d}z}{z}\, z^{\Delta - \bar{h}} (z-1)^c \\ &= \frac{\Gamma(\bar{h} - \Delta - c)}{\Gamma(-c)\Gamma(\bar{h} - \Delta + 1)} - \frac{B_{1/z_{\max}}(\bar{h} - \Delta - c, c+1)}{\Gamma(-c)\Gamma(c+1)} \end{aligned} \tag{16}$$

- The approximation in the first line is twofold. First, we put in a $\bar{z}$-independent upper bound $z_{\max}$ on z, and we put in the t-channel expression for g even though convergence of this expression requires $|z - \beta| < \beta$, which is not going to be true a lot of the time.

- A posteriori, we can justify both approximations since the only part of $S_{c,\Delta}$ which depends on $z_{\max}$ is the incomplete beta-function. If there is a limit in which this term is small in some sense, then we're good to go; the sum might diverge, but we can argue that the divergent part should be suppressed in some other way in this regime.

- As it turns out, such a regime exists: as we let $\bar{h} \gg 1$, we find that the beta function decays as $z_{\max}^{-\bar{h}}$, i.e. it is exponentially suppressed in $\bar{h}$. If Δ is not diverging, then the limit $\bar{h} \to \infty$ is the $J \to \infty$ limit; we should consider the large-spin limit.

- However, before doing that, let's finish setting the scene. The function $a(\Delta, J)$ has poles at the scaling dimensions of physical operators, with the thermal one-point functions as residue. An interesting set of poles that we can immediately identify is located at values of Δ such that $\Delta_\phi - h + m = -n$ (with $n \in \mathbb{N}$), i.e. at

$$\Delta = 2\Delta_\phi + 2n + J \tag{17}$$

These are the scaling dimensions of the double trace operators $[\phi\phi]_{n,J} \sim \phi \partial^{2n} \partial^{\mu_1} \dots \partial^{\mu_J} \phi$.

- In an interacting theory, we'd expect the scaling dimensions to include anomalous dimensions; these turn out to be present, they roughly arise from the sum over k (the story is somewhat subtle and requires a proper treatment of the interchange of the sum over m

and the integrals). For now, we model them by simply declaring that the poles are located at $\Delta = 2\Delta_\phi + 2n + J + 2\delta_n(\bar{h})$, where $2\delta_n(\bar{h})$ is the anomalous dimension; we assume it is fixed in some other way.

- Since δ_n depends on $\bar{h}$ which depends on Δ, this has an effect: it introduces a factor $\mathrm{d}\bar{h}/\mathrm{d}J$ into the residue. Say the pole is located at $\Delta_0 = 2\Delta_\phi + 2n + J + \delta_n(\bar{h}_0)$; then

$$\begin{aligned}\frac{1}{\Delta_\phi + n + \delta_n(\bar{h}) - h} &= \frac{-2}{\Delta - J - 2\Delta_\phi - 2n - 2\left(\delta_n(\bar{h}_0) + \delta_n'(\bar{h}_0)(\bar{h} - \bar{h}_0)\right)} \\ &= -\frac{2}{(1-\delta_n'(\bar{h}_0))(\Delta - \Delta_0)} \\ &= -2\frac{\mathrm{d}\bar{h}}{\mathrm{d}J}\frac{1}{\Delta - \Delta_0} \end{aligned} \tag{18}$$

  where the final equality follows from the fact that $\bar{h}$ is fixed implicitly at the pole, since $\bar{h} = \Delta_\phi + n + J + \delta_n(\bar{h})$.

- So we can get the contribution from $\mathcal{O}$ to the one-point function of $[\phi\phi]_{n,J}$ by taking the residue at the naive pole, but throwing in an additional factor $\mathrm{d}\bar{h}/\mathrm{d}J$. We get[1]

$$\begin{aligned} a^{(\mathcal{O})}_{[\phi\phi]_n}(J) =& -\mathrm{Res}_{\Delta = 2\Delta_\phi + 2n + J}\, a^{(\mathcal{O})}(\Delta, J) \\ =& \frac{a_{\mathcal{O}}(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})}\frac{\mathrm{d}\bar{h}}{\mathrm{d}J}\sum_{m=0}^{n}\sum_{k=0}^{\ell_{\mathcal{O}}}\frac{2(-1)^{n-m}\Gamma(\bar{h}_{\mathcal{O}} - \Delta_\phi - k + 1)}{\Gamma(n-m+1)\Gamma(\bar{h}_{\mathcal{O}} - \Delta_\phi - k + m - n + 1)} \\ & \times 2\pi q_m(J) p_k(\ell_{\mathcal{O}}) S_{h_{\mathcal{O}} - \Delta_\phi + k, \Delta_\phi - m}(\bar{h}) \end{aligned} \tag{19}$$

- So this is the contribution of an operator $\mathcal{O}$ (scaling dimension $\Delta_{\mathcal{O}}$ and spin $\ell_{\mathcal{O}}$) to the one-point function of $[\phi\phi]_{n,J}$.

3 The large-spin expansion

- As mentioned before, our expression is really only valid for $\bar{h} \gg 1$. Luckily, the terms in (19) are already organised as an expansion in $1/\bar{h}$, since

$$S_{c,\Delta}(\bar{h}) = \frac{1}{\Gamma(-c)\bar{h}^{c+1}} + O(\bar{h}^{-c-1}) \tag{20}$$

  while $q_m(J)$ tends to a constant for $J \to \infty$.

- The leading contribution in (19) is therefore the $k = 0$ term:

$$\begin{aligned} a^{(\mathcal{O})}_{[\phi\phi]_n}(J) \approx& \frac{a_{\mathcal{O}}(1+(-1)^J)\Gamma(\ell_{\mathcal{O}} + \frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J+\frac{d-2}{2})\Gamma(\ell_{\mathcal{O}}+1)}\frac{\mathrm{d}\bar{h}}{\mathrm{d}J}\sum_{m=0}^{n}\frac{2(-1)^{n-m}\Gamma(\bar{h}_{\mathcal{O}} - \Delta_\phi + 1)}{\Gamma(n-m+1)\Gamma(\bar{h}_{\mathcal{O}} - \Delta_\phi + m - n + 1)} \\ & \times 2\pi q_m(J) S_{h_{\mathcal{O}} - \Delta_\phi, \Delta_\phi - m}(\bar{h}) \end{aligned} \tag{21}$$

- This further simplifies[2] for the leading double-trace operators $[\phi\phi]_{0,J}$:

$$\begin{aligned} a^{(\mathcal{O})}_{[\phi\phi]_0}(J) &\approx \frac{2a_{\mathcal{O}}(1+(-1)^J)\Gamma(\ell_{\mathcal{O}} + \frac{d-2}{2})\Gamma(J+1)}{\Gamma(J+\frac{d-2}{2})\Gamma(\ell_{\mathcal{O}}+1)}\frac{\mathrm{d}\bar{h}}{\mathrm{d}J} S_{h_{\mathcal{O}} - \Delta_\phi, \Delta_\phi}(\bar{h}) \\ &\approx \frac{2a_{\mathcal{O}}(1+(-1)^J)\Gamma(\ell_{\mathcal{O}} + \frac{d-2}{2})\Gamma(J+1)}{\Gamma(\Delta_\phi - h_{\mathcal{O}})\Gamma(J+\frac{d-2}{2})\Gamma(\ell_{\mathcal{O}}+1)}\frac{\mathrm{d}\bar{h}}{\mathrm{d}J}\frac{1}{\bar{h}^{h_{\mathcal{O}} - \Delta_\phi + 1}} \end{aligned} \tag{22}$$

[1] We usually call the set of operators $[\phi\phi]_{n,J}$ with the same n but different J a double-trace family $[\phi\phi]_n$.

[2] The overall factor of 2 is not present in the paper; it showed up in my derivation of the inversion formula and I haven't yet figured out how to get rid of it.

- So the thermal one-point function of an operator (e.g. $[\phi\phi]_{n,J}$) can be written as a sum of contributions from potentially many operators $\mathcal{O}$; these contributions are all linear in $a_{\mathcal{O}}$, and are naturally organised in expansions in $1/\bar{h}$. Thus, if we consider large-spin operators, we can write down a perturbative expression for their one-point function in terms of the one-point function of other operators, and the order at which these contribute is determined by their twist $h_{\mathcal{O}}$.

- The dominant contribution to the large-spin expansion therefore comes from low-twist operators, primarily the identity (which has zero twist), light scalar operators, and the stress tensor (which has twist $d-2$).

- As an aside, the t-channel OPE is most appropriate when $z \approx \beta$; and we saw that the pole in $a^{(\mathcal{O})}(\Delta, J)$ came about precisely because of the behaviour near $\bar{z} \approx 0$. This corresponds to $\tau \approx \beta/2$ and $t_L \approx \beta/2$, i.e. a point which you can reach along two distinct null geodesics. That is why this is called a 'double-lightcone limit'.

4 Corrections to residues

- So we now know how to relate the one-point functions of double-trace operators to one-point functions of other operators; e.g. the leading $1/\bar{h}$ behaviour of the one-point function of operators in the $[\phi\phi]_0$ family is

$$a_{[\phi\phi]_0}(J) \approx \sum_{\mathcal{O}} \frac{2a_{\mathcal{O}}(1+(-1)^J)\Gamma(\ell_{\mathcal{O}}+\frac{d-2}{2})\Gamma(J+1)}{\Gamma(J+\frac{d-2}{2})\Gamma(\ell_{\mathcal{O}}+1)} \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} S_{h_{\mathcal{O}}-\Delta_\phi,\Delta_\phi}(\bar{h}) \tag{23}$$

- You can then put *this* back into the t-channel representation of the two-point function (recall that the leading $1/\bar{h}$ behaviour comes from $k=0$):

$$\begin{aligned} g(z,\bar{z}) &\supset \sum_{[\phi\phi]_0} \frac{a_{[\phi\phi]_0}(J)}{\beta^\Delta} \frac{\Gamma(J+\frac{d-2}{2})}{\Gamma(J+1)\Gamma(\frac{d-2}{2})} (\beta-z)^{h-\Delta_\phi}(\beta-\bar{z})^{\bar{h}-\Delta_\phi} \\ &\approx \sum_{[\phi\phi]_0}\sum_{\mathcal{O}} \frac{2a_{\mathcal{O}}(1+(-1)^J)\Gamma(\ell_{\mathcal{O}}+\frac{d-2}{2})}{\Gamma(\ell_{\mathcal{O}}+1)\Gamma(\frac{d-2}{2})\beta^\Delta} \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} S_{h_{\mathcal{O}}-\Delta_\phi,\Delta_\phi}(\bar{h})(\beta-z)^{h-\Delta_\phi}(\beta-\bar{z})^{\bar{h}-\Delta_\phi} \end{aligned} \tag{24}$$

  where we stress that parameters without subscript are associated with $[\phi\phi]_{0,J} \in [\phi\phi]_0$.

- So a family of operators (e.g. double-trace operators) contributes to the two-point function; every operator in this family is composed of many other operators, organised by their twist.

- The idea is now to apply the inversion formula *again*, using this expression as input. Then you get the contribution of this particular family of double-trace operators (itself determined by one-point functions of a large set of operators) to the one-point function of any other operator.

- However, it turns out that you cannot just commute the sum over $[\phi\phi]_0$ and the integrals over z and $\bar{z}$, which isn't too unexpected (we're trying to reproduce poles after all), so we need to perform the infinite sum over the family $[\phi\phi]_0$ first.

- This sum is a sum over the spins J of the double-trace family, or equivalently over $\bar{h} = \Delta_\phi + J + \delta_0(\bar{h}) = \bar{h}_f + \delta_0(\bar{h})$ where $\bar{h}_f$ is the $\bar{h}$ one would naively associate with a spin-J member of this particular family. h is fixed as $h = \Delta_\phi + \delta_n(\bar{h}) = h_f + \delta_0(\bar{h})$.

- The sum that we need to perform is therefore

$$\sum_{\bar{h}=\bar{h}_f+\delta_0(\bar{h})} \left(1+(-1)^{\bar{h}-h_f-\delta_0(\bar{h})}\right) \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} S_{h_\mathcal{O}-\Delta_\phi,\Delta_\phi}(\bar{h}) \left(1-\frac{z}{\beta}\right)^{h_f+\delta_0(\bar{h})-h_e} \left(1-\frac{\bar{z}}{\beta}\right)^{\bar{h}-h_e} \tag{25}$$

  where $h_e = \Delta_\phi$ the twist of the 'external' operator (i.e. ϕ).

- The trick to evaluating such a sum is to find that for large $\bar{h}$, the anomalous dimension is very small. Let's focus on the sum starting at some sufficiently large $\bar{h}_0$, which may indeed be all there is since we were already considering the large-spin members of the $[\phi\phi]_0$ family. Then you can Taylor expand $(1-z)^{\delta_0(\bar{h})}$ (setting $z/\beta \to z$ and focusing on the term with a definite sign for the moment):

$$\sum_{m=0}^{\infty} \frac{\ln^m(1-z)}{m!} (1-z)^{h_f-h_e} \sum_{\bar{h}=\bar{h}_f+\delta_0(\bar{h})} \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} \delta_0^m(\bar{h}) S_{h_\mathcal{O}-\Delta_\phi,\Delta_\phi}(\bar{h}) (1-\bar{z})^{\bar{h}-h_e} \tag{26}$$

- This makes clearer why we cannot commute the sum over $\bar{h}$ and the integral over $\bar{z}$; if you expand around $\bar{z}=0$, you get integrals that you can compute but they depend exponentially on $\bar{h}$, making the sum divergent.

- What kind of $\bar{z}$ dependence should we expect? From the inversion formula, we know that there have to be contributions that depend on integer powers of $\bar{z}$; after all, these gave us the contributions of any operator to the double-trace families to begin with.

- All the other operators in the spectrum should therefore correspond to contributions that do not scale as a non-negative integer power of $\bar{z}$. That is, we expect

$$\sum_{\bar{h}=\bar{h}_f+\delta_0(\bar{h})} \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} f(\bar{h}) (1-\bar{z})^{\bar{h}-h_e} = \sum_{a\in A} c_a \bar{z}^a + \sum_{k=0}^{\infty} \alpha_k \bar{z}^k \tag{27}$$

  with $A = \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$.

- A slightly more rigorous way to see this is to write the left-hand side as a contour integral and see that terms that are singular (i.e. scale with $\bar{z}^a$, with $a \in A$) come from the asymptotic behaviour of the integrand, which is independent of $\delta_0(\bar{h})$; since we can actually do the integral for $\delta_0(\bar{h}) = 0$, this means that we can find these singular terms.

- The regular terms follow from a very similar calculation.

- Besides the definite-sign terms, there is also an alternating contribution. It is actually convergent if you first do the $\bar{z}$ integral and then do the sum over $\bar{h}$, so you only get the power law behaviour:

$$\sum_{\bar{h}=\bar{h}_f+\delta_0(\bar{h})} (-1)^J \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} f(\bar{h}) (1-\bar{z})^{\bar{h}-h_e} = \sum_{k=0}^{\infty} \alpha_k^- \bar{z}^k \tag{28}$$

- In total, we get that

$$\sum_{[\phi\phi]_0} \left(1+(-1)^J\right) \frac{\mathrm{d}\bar{h}}{\mathrm{d}J} S_{h_\mathcal{O}-\Delta_\phi,\Delta_\phi}(\bar{h}) (1-z)^{h-\Delta_\phi} (1-\bar{z})^{\bar{h}-\Delta_\phi} \tag{29}$$

can be expressed as

$$\sum_{m=0}^{\infty} \ln^m(1-z)(1-z)^{h_f-h_e}\left(\sum_{a\in A} c_a[\delta_0^m S_{h_{\mathcal{O}}-\Delta_\phi,\Delta_\phi}/m!]\bar{z}^a + \sum_{k=0}^{\infty}\alpha_k^{\text{even}}[\delta_0^m S_{h_{\mathcal{O}}-\Delta_\phi,\Delta_\phi}/m!]\bar{z}^k\right) \tag{30}$$

where $\alpha_k^{\text{even}} = \alpha_k + \alpha_k^-$.

- What happens when you put this back into the inversion formula[3]? First, observe that $\ln^m(1-z)(1-z)^{h_f-h_e} = \partial_{h_f}^m(1-z)^{h_f-h_e}$. This means that the regular part of the sum (depending on integer powers of $\bar{z}$) will, just as before, represent contributions to the residue at the poles of double-trace operators.
- Particularly interesting is that you can consider the leading contribution ($k=0$) of the $[\phi\phi]_0$ family to operators *in that very family*. The result is

$$\begin{aligned} a_{[\phi\phi]_0}(J) = \left(1+(-1)^J\right)\frac{\mathrm{d}\bar{h}}{\mathrm{d}J}\sum_{\mathcal{O}}\frac{a_{\mathcal{O}}\Gamma(J+1)\Gamma(\ell_{\mathcal{O}}+\frac{d-2}{2})}{\Gamma(\ell_{\mathcal{O}}+1)\Gamma(J+\frac{d-2}{2})}\Big(S_{h_{\mathcal{O}}-\Delta_\phi,\Delta_\phi}(\bar{h}) \\ + \sum_{m=0}^{\infty}\alpha_0^{\text{even}}[\delta_0^m S_{h_{\mathcal{O}}-\Delta_\phi,\Delta_\phi}/m!]S_{0,\Delta_\phi}^{(m)}(\bar{h})\Big) \end{aligned} \tag{31}$$

  The first line was already there before; the second line is a correction because operators in the $[\phi\phi]_0$ family contribute to other operators in that family as well. Overall, the above formula represents the following:

  1. Many operators contribute to the one-point function of $[\phi\phi]_{0,J}$, and we organise those operators by increasing twist;
  2. The operators in the $[\phi\phi]_0$ family in turn contribute to the part any operator $\mathcal{O}$ plays in the two-point function (switching the sums in (24));
  3. These operators $\mathcal{O}$ again contribute to the one-point function of $[\phi\phi]_{0,J}$.
- You can either go through this process repeatedly or treat it as a fixed-point problem to get the full self-corrected one-point function.

5 An 'example': the $d = 3$ Ising model

- An instance in which we can apply these techniques is in the $d = 3$ Ising model, where the low-twist spectrum is known through the conformal bootstrap.
- The Ising model has two relevant operators (with $\Delta < 3$): σ and ϵ. They are scalars which are respectively odd and even under the $\mathbb{Z}_2$ symmetry of the Ising model, with

$$\Delta_\sigma \approx 0.51 \qquad\qquad \Delta_\epsilon = 1.41 \tag{32}$$

- We can for example study the $\langle\sigma\sigma\rangle_\beta$ correlator, for which we need the lowest-twist contributions to the $\sigma\times\sigma$ OPE:

$$\sigma\times\sigma = \mathbb{1} + \epsilon + T + \sum_{\ell=2}^{\infty}[\sigma\sigma]_{0,2\ell} + \dots \tag{33}$$

  at least schematically (there should also be OPE coefficients).

[3] As a reminder, this would give you the contribution of the double-trace family $[\phi\phi]_0$ to the one-point function of whichever operator you're trying to analyse using the inversion formula.

- Therefore, the leading contributions to the $[\sigma\sigma]_{0,J}$ one-point functions are

$$a_{[\sigma\sigma]_0}(J) = \left(1+(-1)^J\right)\frac{\mathrm{d}\bar{h}}{\mathrm{d}J}\sum_{\mathcal{O}=\mathbb{1},\epsilon,T}\frac{a_{\mathcal{O}}\Gamma(J+1)\Gamma(\ell_{\mathcal{O}}+\frac{1}{2})}{\Gamma(\ell_{\mathcal{O}}+1)\Gamma(J+\frac{1}{2})}S_{h_{\mathcal{O}}-\Delta_\sigma,\Delta_\sigma}(\bar{h}) \tag{34}$$

- Recall that, at large $\bar{h}$, $S_{c,\Delta}(\bar{h}) \sim \bar{h}^{-c-1}$, and since we are considering a large-J expansion we have $\Gamma(J+1)/\Gamma(J+\frac{1}{2}) \approx \sqrt{J}$, so

$$a_{[\sigma\sigma]_0}(J) \approx \left(1+(-1)^J\right)\left(\frac{1.04}{J^{\frac{1}{2}-\Delta_\sigma}} + \frac{0.01a_T}{J^{1-\Delta_\sigma}} - \frac{0.29a_\epsilon}{J^{\frac{1}{2}+\frac{\Delta_\epsilon}{2}-\Delta_\sigma}}\right) \tag{35}$$

- The next set of operators in the $\sigma \times \sigma$ OPE is the $[\sigma\sigma]_0$ family, and so we need the anomalous dimension of this family. It was computed using the $T = 0$ bootstrap:

$$\delta_0(\bar{h}) \approx -\frac{0.0014}{\bar{h}} \tag{36}$$

  You can then use this in (31) and either iterate or solve the eigenvalue problem to find the fixed point. The result is e.g.

$$a_{[\sigma\sigma]_{0,4}} \approx 2.1 - 0.22a_\epsilon + 0.010a_T \tag{37}$$

- The strategy to complete the bootstrap is (very roughly) as follows. You compute the self-corrected one-point function $a_{[\sigma\sigma]_0}(J)$ using a_ϵ and a_T as unknowns. Then you impose that $a_T^{([\sigma\sigma])} = a_{[\sigma\sigma]_0}^{([\sigma\sigma])}(J=2)$, i.e. that the stress tensor is the spin-2 operator in the $[\sigma\sigma]_0$ family. This allows you to write the correlator $\langle\sigma\sigma\rangle_\beta$ as a function of a single unknown, a_ϵ. You can find it by imposing the KMS condition.

6 Further reading

- These notes are based on section 6 in The Conformal Bootstrap at Finite Temperature.
- Bootstrapping the 3d Ising model at finite temperature is a follow-up paper on the $d = 3$ Ising model, which briefly reviews the basic idea of the bootstrap and goes into much more detail on the bootstrapping algorithm.
- A more analytic approach is presented in The analytic bootstrap at finite temperature, which also briefly touches on the Ising model.