Examples with the thermal inversion formula

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1 Recap of last time

- Last time, we introduced the thermal inversion formula. We started with the thermal block decomposition of the propagator, which can be written as a spectral integral:

$$g(\tau,\mathbf{x}) = \frac{1}{\beta^{\Delta_\phi}} \sum_{J=0}^{\infty} \int_{-\infty-\epsilon}^{i\infty-\epsilon} \frac{\mathrm{d}\Delta}{2\pi i} \frac{a(\Delta,J) C_J^{(\frac{d-2}{2})}(\eta)}{(|x|/\beta)^{2\Delta_\phi-\Delta}} \tag{1}$$

  where $a(\Delta, J)$ is any function which only has poles that look like

$$a(\Delta,J) \sim -\frac{a_{\mathcal{O}}}{\Delta-\Delta_{\mathcal{O}}} \tag{2}$$

  with $a_{\mathcal{O}}$ (closely related to) the thermal one-point coefficient, $|x|^2 = \tau^2 + \mathbf{x}^2$, and $\eta = \tau/|x|$. The idea of the thermal bootstrap is to impose the KMS condition as a non-trivial constraint on g and extract the $a_{\mathcal{O}}$; the thermal inversion formula helps by extracting one-point data from the two-point function.

- By manipulating the two-point function in a variety of ways, we could find that

$$a(\Delta,J) = \frac{\beta^{2\Delta_\phi-d}}{N_J} \int_{|x|<\beta} \mathrm{d}^d x\, C_J^{(\frac{d-2}{2})}(\eta) g(\tau,\mathbf{x}) (|x|/\beta)^{2\Delta_\phi-\Delta-d} \tag{3}$$

  This is called the 'Euclidean inversion formula' since you integrate over a Euclidean sphere.

- Then we defined complex coordinates, with $r = |\mathbf{x}|$:

$$z = \tau + ir = |x|w \qquad \bar{z} = \tau - ir = \frac{|x|}{w} \tag{4}$$

  This allowed us to recast the Euclidean integral as a two-dimensional integral (g only depends on τ and r); the w integral was a contour integral.

- Assuming that the two-point function is polynomially bounded at large w (no proof, analogous to Regge limit in the $T = 0$ four-point function) and only has branch cuts along the real axis (with intervals of analyticity in $[-\beta/|x|, -|x|/\beta]$ and $[|x|/\beta, \beta/|x|]$; this can be argued for a posteriori), we can deform the contour of the w integral to only take contributions from the discontinuity along the positive real w axis.

- Writing the result in terms of z and $\bar{z}$, we conclude that

$$a(\Delta,J) = \frac{(1+(-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)\beta^{\Delta}}{2\pi\Gamma(J+\frac{d-2}{2})} \int_0^{\beta} \mathrm{d}\bar{z} \int_{\beta}^{\beta^2/\bar{z}} \mathrm{d}z (z\bar{z})^{\Delta_\phi - \frac{\Delta+d}{2}} (z-\bar{z})^{d-2} F_J\left(\sqrt{\bar{z}/z}\right) \\ \times \mathrm{Disc}[g(z,\bar{z})] \tag{5}$$

where the function F_J is defined as

$$F_J(x) = x^{J+d-2}{}_2F_1\left(J+d-2, \frac{d}{2}-1, J+\frac{d}{2}, x^2\right) \tag{6}$$

The expression for a in terms of z and $\bar{z}$ is called a 'Lorentzian inversion formula' because z and $\bar{z}$ are real, implying that we have Wick rotated $r \to -ir = x_L$. This gives a Lorentzian but Kaluza-Klein compactified manifold.

- Iliesu and collaborators claim that the discontinuity can be taken across the positive real axis of the complex z plane, keeping $\bar{z}$ real. On the other hand, the original discontinuity was in w, so we might expect

$$\mathrm{Disc}[g(z,\bar{z})] = \frac{1}{i}\lim_{\epsilon\downarrow 0}\left(g(z+i\epsilon, \bar{z}-i\epsilon) - g(z-i\epsilon, z+i\epsilon)\right) \tag{7}$$

- Heuristically, we could think of Iliesu and collaborators' procedure as reflecting the fact that poles in $a(\Delta, J)$ are mostly coming from the $\bar{z} \approx 0$ region; after all, the integral over z only covers a region in which the integrand is finite, whereas $\bar{z}$ actually goes near points where the integrand blows up[1]. Together with the $\bar{z}$-dependence of the integration bound of the z-integral, this suggests that we should first evaluate the z integral, and thus care more[2] about the discontinuity as a function of z.

2 The mean field two-point function

- As a first check, we can look at mean field theory (MFT). MFT is like free field theory, but the field ϕ can have any scaling dimension Δ_ϕ (instead of only $\Delta_\phi = \frac{d-2}{2}$).

- We can actually obtain an explicit expression for the thermal two-point function or propagator in this theory.

- The idea is that the main difference between a thermal two-point function and a non-thermal two-point function is that the former is periodic in τ while the latter is not; this is the statement of the KMS condition, and of the notion of a thermal cylinder.

- For any periodic function, you can write

$$g(\tau, \mathbf{x}) = \sum_{n\in\mathbb{Z}} g(\tau - \beta n, \mathbf{x}) \tag{8}$$

where the g on the left-hand side is taken to have support for all $\tau \in \mathbb{R}$ but the g in the sum on the right-hand side is taken to only have support when the first argument falls within $[0, \beta)$. Physically, this says that you can go from $(0, \mathbf{0})$ to $(\tau, \mathbf{x})$ by going there directly or by wrapping around the cylinder in the τ-direction any number of times.

- Usually this doesn't help much: you've replaced a complicated object by an infinite sum of equally complicated objects. But for a free field (like an MFT), nothing will happen between the two insertions, no matter how often you wrap around the cylinder.

[1] This implies, roughly speaking, that the $\bar{z}$ integrals determines where poles in $a(\Delta, J)$ occur, while the z integral determines the residue at these poles.

[2] This argument is shaky at best and wrong at worst. We couldn't quite come up with a stronger argument besides 'if you look at the integrals sufficiently carefully, this should be what you're supposed to do'. Presumably, the choice for where to put the discontinuity was made when we changed variables from $|x|$ and w to z and $\bar{z}$. What is the right way to find the appropriate branch cuts?

- This implies that you can replace the individual terms on the right (which are not *required* to be periodic, the periodicity is enforced by the sum) by the $T = 0$ propagator; after all, compactifying a single direction on the plane does not bring you to a geometrically different space[3].

- This expression is called the 'method of images' and you can use it to do thermal CFT perturbatively. For the MFT, we know the $T = 0$ position space propagator:

$$g(\tau, \mathbf{x}) = \sum_{n \in \mathbb{Z}} \frac{1}{[(\tau - n\beta)^2 + r^2]^{\Delta_\phi}} = \sum_{n \in \mathbb{Z}} \frac{1}{[(n\beta - z)(n\beta - \bar{z})]^{\Delta_\phi}} \tag{9}$$

- This does not immediately look like the thermal block decomposition, but it can be rewritten. Consider the following generating function for the Gegenbauer polynomials:

$$\frac{1}{|n\beta|^{2\Delta_\phi}} \frac{1}{[1 - 2\frac{\tau}{|x|}\frac{|x|}{n\beta} + \frac{|x|^2}{n^2\beta^2}]^{\Delta_\phi}} = \frac{1}{\beta^{2\Delta_\phi}} \sum_{j=0}^{\infty} \frac{\mathrm{sgn}(n^j)}{|n|^{2\Delta_\phi + j}} C_j^{(\Delta_\phi)}(\eta)(|x|/\beta)^j \tag{10}$$

- You can put this back into the expression for the two-point function and find

$$\begin{aligned} g(\tau, \mathbf{x}) &= \frac{1}{\beta^{2\Delta_\phi}} \frac{1}{(|x|/\beta)^{2\Delta_\phi}} + \frac{1}{\beta^{2\Delta_\phi}} \sum_{j=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1 + (-1)^j}{n^{2\Delta_\phi + j}} \right) C_j^{(\Delta_\phi)}(\eta)(|x|/\beta)^j \\ &= \frac{1}{\beta^{2\Delta_\phi}} \frac{1}{(|x|/\beta)^{2\Delta_\phi}} + \frac{2}{\beta^{2\Delta_\phi}} \sum_{j'=0}^{\infty} \zeta(2\Delta_\phi + 2j') C_{2j'}^{(\Delta_\phi)}(\eta)(|x|/\beta)^{2j'} \end{aligned} \tag{11}$$

where we replaced $j \to 2j'$ to account for the fact that $1 + (-1)^j$ kills terms with odd j.

- The Gegenbauer polynomial is unfortunately the wrong one, so we need another expansion:

$$C_{2j'}^{(\Delta_\phi)}(\eta) = \sum_{\ell=0}^{j'} \frac{(2\ell + \frac{d-2}{2})(\Delta_\phi)_{j'+\ell}(\Delta_\phi - \frac{d-2}{2})_{j'-\ell}}{\Gamma(j' - \ell + 1)(\frac{d-2}{2})_{j'+\ell+1}} C_{2\ell}^{(\frac{d-2}{2})}(\eta) \tag{12}$$

- Using this identity and switching the ℓ and j' sums, we get that

$$\begin{aligned} g(\tau, \mathbf{x}) = &\frac{1}{\beta^{2\Delta_\phi}} \frac{1}{(|x|/\beta)^{2\Delta_\phi}} \\ &+ \frac{1}{\beta^{2\Delta_\phi}} \sum_{\ell=0}^{\infty} \sum_{j'=\ell}^{\infty} \frac{2\zeta(2\Delta_\phi + 2j')(2\ell + \frac{d-2}{2})(\Delta_\phi)_{j'+\ell}(\Delta_\phi - \frac{d-2}{2})_{j'-\ell}}{\Gamma(j' - \ell + 1)(\frac{d-2}{2})_{j'+\ell+1}} \frac{|x|^{2j'}}{\beta^{2j'}} C_{2\ell}^{(\frac{d-2}{2})}(\eta) \end{aligned} \tag{13}$$

- We can now replace $j' \to n + \ell$ and sum over $n \in \mathbb{N}_0$ instead of j'; finally this results in

$$\begin{aligned} g(\tau, \mathbf{x}) = &\frac{1}{\beta^{2\Delta_\phi}} \frac{1}{(|x|/\beta)^{2\Delta_\phi}} \\ &+ \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{2\zeta(2\Delta_\phi + 2n + 2\ell)(2\ell + \frac{d-2}{2})(\Delta_\phi)_{n+2\ell}(\Delta_\phi - \frac{d-2}{2})_n}{\beta^{2\Delta_\phi}\Gamma(n+1)(\frac{d-2}{2})_{n+2\ell+1}} \frac{|x|^{2n+2\ell}}{\beta^{2n+2\ell}} C_{2\ell}^{(\frac{d-2}{2})}(\eta) \end{aligned} \tag{14}$$

[3] Think of a sheet of paper: it is flat because parallel lines never meet/Pythagoras' theorem takes the familiar form/angles inside triangles add up to 180° or any other reason. Now roll the sheet into a cylinder; all diagrams you drew to verify that the sheet was flat are still the same, so the cylinder is flat as well.

- This *is* exactly of the form we would expect; it implies that the only operators which contribute to the (thermal) $\phi \times \phi$ OPE are the identity (with $\Delta_{\mathbb{1}} = 0$ and $a_{\mathbb{1}} = 1$) and the so-called double-trace operators $[\phi\phi]_{n,2\ell}$ with

$$\Delta_{[\phi\phi]_{n,2\ell}} = 2\Delta_\phi + 2n + 2\ell \tag{15}$$

$$a_{[\phi\phi]_{n,2\ell}} = \frac{2\zeta(2\Delta_\phi + 2n + 2\ell)(2\ell + \frac{d-2}{2})(\Delta_\phi)_{n+2\ell}(\Delta_\phi - \frac{d-2}{2})_n}{\Gamma(n+1)(\frac{d-2}{2})_{n+2\ell+1}} \tag{16}$$

- They are called double-trace because you can schematically write them as

$$[\phi\phi]_{n,2\ell} = \phi\partial^{\mu_1}\cdots\partial^{\mu_{2\ell}}\partial^{2n}\phi - \text{traces} \tag{17}$$

  Observe how these are all even-spin operators.

- We can also consider the discontinuity. Taking Iliesu's approach, we see that

$$\begin{aligned} \text{Disc}\left[\frac{1}{(n\beta - z)^{\Delta_\phi}(n\beta - \bar{z})^{\Delta_\phi}}\right] &= \frac{-i}{(n\beta - \bar{z})^{\Delta_\phi}} \lim_{\epsilon \downarrow 0}\left(\frac{1}{(n\beta - z - i\epsilon)^{\Delta_\phi}} - \frac{1}{(n\beta - z + i\epsilon)^{\Delta_\phi}}\right) \\ &= \frac{-i}{(n\beta - \bar{z})^{\Delta_\phi}} \frac{1 - e^{-2\pi\Delta_\phi i}}{(n\beta - z)^{\Delta_\phi}}\Theta(z - n\beta) \\ &= \frac{2\sin(\pi\Delta_\phi)\Theta(z - n\beta)}{(n\beta - \bar{z})^{\Delta_\phi}(z - n\beta)^{\Delta_\phi}} \end{aligned} \tag{18}$$

  where we used the fact that the complex conjugate of (a limit to) a real number always differs from the original by a phase of 2π; the Heaviside comes about because we put the branch cut of a root along the negative real axis.

- The naive approach would produce a phase factor that's twice as large, so we cannot claim them to be equivalent.

3 Verifying the inversion formula

- We can now put the discontinuity back into the inversion formula to get an integral which we cannot solve. Up to some constants and assuming even J, a term with $n > 0$ contributes

$$\int_0^\beta \mathrm{d}\bar{z} \int_{n\beta}^{\beta \max(n, \beta/\bar{z})} \mathrm{d}z\, F_J\left(\sqrt{\bar{z}/z}\right) \frac{(z\bar{z})^{\Delta_\phi - \frac{\Delta+d}{2}}(z - \bar{z})^{d-2}}{(n\beta - \bar{z})^{\Delta_\phi}(z - n\beta)^{\Delta_\phi}} \tag{19}$$

- If indeed the contribution from $\bar{z} \approx 0$ is what dominates this expression, then we can replace the upper bound for z by ∞, exchange the z and $\bar{z}$ integrals, and rescale $\bar{z} \to z\bar{z}$:

$$\int_{n\beta}^\infty \mathrm{d}z \int_0^{\beta/z} z\mathrm{d}\bar{z}\, F_J\left(\sqrt{\bar{z}}\right) \frac{(z^2\bar{z})^{\Delta_\phi - \frac{\Delta+d}{2}} z^{d-2}(1 - \bar{z})^{d-2}}{(n\beta - z\bar{z})^{\Delta_\phi}(z - n\beta)^{\Delta_\phi}} \tag{20}$$

- Again, if the main contribution comes from $\bar{z} = 0$, then we can take the upper bound of the $\bar{z}$ integral to be 1 (recall that $\bar{z}$ is now dimensionless); we can also make z dimensionless by rescaling $z \to n\beta z$:

$$\frac{1}{(n\beta)^\Delta} \int_0^1 \mathrm{d}\bar{z} \int_1^\infty \mathrm{d}z\, F_J(\sqrt{\bar{z}}) \frac{(z^2\bar{z})^{\Delta_\phi - \frac{\Delta+d}{2}} z^{d-1}(1 - \bar{z})^{d-2}}{(1 - z\bar{z})^{\Delta_\phi}(z - 1)^{\Delta_\phi}} \tag{21}$$

- You can actually recognise the z integral as another representation of the ${}_2F_1$ hypergeometric function:

$$\frac{\Gamma(1-\Delta_\phi)\Gamma(\Delta-\Delta_\phi)}{(n\beta)^\Delta\Gamma(1+\Delta-2\Delta_\phi)}\int_0^1 \mathrm{d}\bar{z}\, F_J(\sqrt{\bar{z}})\bar{z}^{\Delta_\phi-\frac{\Delta+d}{2}}(1-\bar{z})^{d-2}{}_2F_1\left(\Delta_\phi, 2\Delta_\phi-\Delta, \Delta_\phi+1-\Delta, \bar{z}\right) \tag{22}$$

  Iliesu and collaborators do not consider the case that $n<0$; or rather, they seem to claim that it contributes the same as $n>0$. The precise argument for doing so is not immediately clear to us.

- The final step is to expand the integrand in powers of $\bar{z}$ and identify poles in Δ.