The Index Map

idea

$$\begin{tikzcd} K_1(I) \arrow[r, "K_1(\varphi)"] & K_1(A) \arrow[r, "K_1(\psi)"] & K_1(B) \arrow[dll, "\delta_1"'] \\ K_0(I) \arrow[r, "K_0(\varphi)"'] & K_0(A) \arrow[r, "K_0(\psi)"'] & K_0(B) \end{tikzcd}$$

we want to find δ_1 that turns this sequence to an exact one

lem given a SES $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$, $u \in \mathcal{U}_n(\tilde{B})$, then

1) $\exists v \in \mathcal{U}_{2n}(\tilde{A})$, $p \in P_{2n}(\tilde{I})$ s.t. $\psi(v) = u \oplus u^*$, $\varphi(p) = v(1_n \oplus 0)v^*$, $s(p) = 1_n \oplus 0$

2) if $w \in \mathcal{U}_{2n}(\tilde{A})$, $q \in P_{2n}(\tilde{I})$ satisfy $\psi(w) = u \oplus u^*$, $\varphi(q) = w(1_n \oplus 0_n)w^*$, then $s(q) = 1_n \oplus 0_n$ and $p \sim_u q$ in $P_{2n}(\tilde{I})$

pf 1) ψ surjective $\Rightarrow \exists v \in \mathcal{U}_{2n}(\tilde{A})$ s.t. $\psi(v) = u \oplus u^*$, and so $\psi(v(1_n \oplus 0_n)v^*) = 1_n \oplus 0_n$, which is a scalar element, hence $\exists p \in M_{2n}(\tilde{I})$ s.t. $\varphi(p) = v(1_n \oplus 0_n)v^*$; note that p is a projection and $\psi \circ \varphi(p) = 1_n \oplus 0_n$, a scalar element, so $s(p) = 1_n \oplus 0_n$ (by naturality)

2) by a similar argument we get $s(q) = 1_n \oplus 0_n$, then since $\psi(wv^*) = 1$, $\exists z \in M_{2n}(\tilde{I})$ s.t. $\varphi(z) = wv^*$, and since φ is injective, z is unitary; finally, $\varphi(zpz^*) = \varphi(q) \Rightarrow zpz^* = q$, i.e. $p \sim_u q$ $\square$

def $\mu : \mathcal{U}_\infty(\tilde{B}) \to K_0(I)$ by $\mu(u) = [p]_0 - [s(p)]_0$, where $p \in P_\infty(\tilde{I})$ is as in the lemma before

lem 1) $\mu(u \oplus v) = \mu(u) + \mu(v)$

2) $\mu(1) = 0$

3) $u \sim_h v \Rightarrow \mu(u) = \mu(v)$

4) $\mu(\operatorname{im} \psi) = 0$

5) $K_0(\varphi)(\mu(u)) = 0$

def the index map $\delta_1 : K_1(B) \to K_0(I)$ is the unique group hom induced by $\mu : \mathcal{U}_\infty(\tilde{B}) \to K_0(I)$ s.t. $\delta_1([p]_1) = \mu(p)$

prop 1) $\delta_1 \circ K_1(\psi) = 0$

2) $K_0(\varphi) \circ \delta_1 = 0$

prop index map is natural: given a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & 0 \\ 0 \arrow[r] & I' \arrow[r, "\varphi'"'] & A' \arrow[r, "\psi'"'] & B' \arrow[r] & 0 \end{tikzcd}$$

and index maps $\delta_1 : K_1(B) \to K_0(I)$, $\delta_1' : K_1(B') \to K_0(I')$, then

$$\begin{tikzcd} K_1(B) \arrow[r, "\delta_1"] \arrow[d, "K_1(\beta)"'] \arrow[dr, phantom, "\circlearrowleft"] & K_0(I) \arrow[d, "K_0(\gamma)"] \\ K_1(B') \arrow[r, "\delta_1'"'] & K_0(I') \end{tikzcd}$$

pf let $u \in \mathcal{U}_\infty(\tilde{B})$, $\delta_1([u]_1) = [p]_0 - [s(p)]_0$, $\delta_1'([\beta(u)]_1) = [p']_0 - [s(p')]_0$, then

$$\begin{aligned} \delta_1' \circ K_1(\beta)([u]_1) &= \delta_1'([\beta(u)]_1) \\ &= [p']_0 - [s(p')]_0 \\ &= [\gamma(p)]_0 - [s(\gamma(p))]_0 \\ &= K_0(\gamma)([p]_0 - [s(p)]_0) \\ &= K_0(\gamma) \circ \delta_1([u]_1). \end{aligned}$$

∎

lem let $\varphi : A \to B$ be a unital $*$-hom, if φ is surjective, then $\forall u \in \mathcal{U}(B)$, $\exists v \in M_2(A)$ a partial isometry s.t. $\varphi(v) = u \oplus 0$.

prop given a SES $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$, $u \in \mathcal{U}_n(\tilde{B})$, $v \in M_m(\tilde{A})$ a partial isometry s.t. $\psi(v) = u \oplus 0$, then $\exists p, q \in P_m(\tilde{I})$ s.t. $\varphi(p) = 1 - v^*v$, $\varphi(q) = 1 - vv^*$, and the index map is given by $\delta_1([u]_1) = [p]_0 - [q]_0$

Prop let $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$ be a SES with ψ unital,

let $\bar{\varphi}: \tilde{I} \to A$ be the $*$-hom given by $\bar{\varphi}(x + \alpha 1_{\tilde{I}}) = \varphi(x) + \alpha 1_A$, let $u \in \mathcal{U}_n(B)$

1) if $\exists v \in \mathcal{U}_{2n}(A)$, $p \in \mathcal{P}_{2n}(\tilde{I})$ s.t. $\bar{\varphi}(p) = v(1_n \oplus 0_n)v^*$, $\psi(v) = u \oplus u^*$,

then $\delta_1([u]_1) = [p]_0 - [s(p)]_0$

2) $v \in M_m(A)$ partial isometry s.t. $\psi(v) = u \oplus 0_{m-n}$, then $\bar{\varphi}(p) = 1 - v^*v$

and $\bar{\varphi}(q) = 1 - vv^*$ for some $p, q \in \mathcal{P}_m(\tilde{I})$, and $\delta_1([u]_1) = [p]_0 - [q]_0$

pf 1) let $f_n = 1_{M_n(\tilde{A})} - 1_{M_n(A)}$, ~~[illegible]~~ $g_n = 1_{M_n(\tilde{B})} - 1_{M_n(B)}$, then

$u' = u + g_n \in \mathcal{U}_n(\tilde{B})$ and $v' = v + f_n \in \mathcal{U}_n(\tilde{A})$, also $\tilde{\psi}(v') = u' \oplus (u')^*$;

note that $\psi \circ \bar{\varphi} \circ s(p) = \psi \circ \bar{\varphi}(p) = 1_{M_n(B)} \oplus 0$, hence $s(p) = 1_{M_n(\tilde{I})} \oplus 0$ and

$\bar{\varphi} \circ s(p) = 1_{M_n(A)} \oplus 0$, therefore

$$\tilde{\varphi}(p) = \varphi(p - s(p)) + \tilde{\varphi}(s(p)) = \bar{\varphi}(p) - \bar{\varphi}(s(p)) + \tilde{\varphi}(s(p)) = \bar{\varphi}(p) + f_n \oplus 0 = v'(1_{M_n(\tilde{A})} \oplus 0)(v')^*.$$

therefore $\delta_1([u']_1) = [p]_0 - [s(p)]_0$ and $[u']_1 = [u]_1$. ◻

lem $\ker \delta_1 \subset \operatorname{im} K_1(\psi)$ $[I \triangleleft A]$

pf let $[u]_1 \in \ker \delta_1$ and note $\exists w_1 \in M_{2n}(\hat{A})$ partial isometry with

$\psi(w_1) = u \oplus 0_n$, hence $0 = \delta_1([u]_1) = [1 - w_1^* w_1]_0 - [1 - w_1 w_1^*]_0$, so

$1 - w_1^* w_1 \sim_s 1 - w_1 w_1^* \Rightarrow \overset{w_2^* w_2}{\cancel{w_2}} = (1 - w_1^* w_1) \oplus 1_k \sim_0 (1 - w_1 w_1^*) \oplus 1_k = w_2 w_2^*$,

for $w_2 \in M_m(\tilde{I})$ partial isometry, now

$$\psi(w_2^* w_2) = \psi(w_2 w_2^*) = 0_n \oplus 1_{m-n}$$

and $\psi(w_2)$ is a scalar matrix because $w_2 \in M_n(\tilde{I})$ (?), hence $w_2 \sim_h 0_n \oplus 1_{m-n}$;

setting $v = (w_1 \oplus 0) + w_2$ yields $\psi(v) \sim_h u \oplus 1_{m-n}$, and so

$$[u]_1 = [\psi(v)]_1 = K_1(\psi)([v]_1).$$ ◻

lem $\ker K_0(\varphi) \subset \operatorname{im} \delta_1$

pf write $g = [p]_0 - [s(p)]_0$, $wpw^* = s(p)$ for $g \in \ker K_0(\varphi)$, $p \in \mathcal{P}_n(\hat{I})$, $w \in \mathcal{U}_n(\tilde{A})$; let

$u_0 = \psi(w(1-p))$ be a partial isometry, then

$$1 - u_0^* u_0 = \psi(p) = \psi(s(p)) = 1 - u_0 u_0^*.$$

then $u = u_0 + (1 - u_0^* u_0) \in \mathcal{U}_n(\tilde{B})$; let $v_1 = w(1-p) \oplus s(p)$, then $\psi(v_1) = u_0 \oplus s(p)$; let $z = \begin{pmatrix} 1-s(p) & s(p) \\ s(p) & 1-s(p) \end{pmatrix}$ and $v = z v_1 z^*$, then

$\psi(v) = z\,\psi(v_1)\,z^* = z(u_0 \oplus s(p))z^* = u \oplus 0$, and so

$$\begin{aligned} \delta_1([u]_1) &= [1 - v^* v]_0 - [1 - v v^*]_0 \\ &= [1 - v_1^* v_1]_0 - [1 - v_1 v_1^*]_0 \\ &= [p \oplus (1 - s(p))]_0 - [s(p) \oplus (1 - s(p))]_0 \\ &= g. \end{aligned}$$

$\square$

<u>def</u> H infinite-dim separable Hilbert space, $T \in B(H)$ is a Fredholm operator if $\operatorname{im} T$ is closed, $\dim \ker T < \infty$, $\dim \ker T^* < \infty$.

<u>prop</u> $T \in B(H)$ Fredholm, then $\operatorname{ind} T = \dim \ker T - \dim \ker T^* = (K_0(\mathrm{Tr}) \circ \delta_1)([\pi(T)]_1)$, where $\pi : B(H) \to B(H)/K$, $K \subset B(H)$ compact operators.

<u>ex</u> $K_1(B(H)/K) \cong K_0(K) \cong \mathbb{Z}$

<u>def</u> $SA = C_0((0,1), A)$

<u>lem</u> X locally cpct Hdff, A C^*-algebra, $f \in C_0(X)$, $a \in A$, then $\qquad fa(x) = f(x)a$

$$\operatorname{span}\{ fa \mid f \in C_0(X),\ a \in A\} \subset C_0(X, A)$$

is dense

<u>prop</u> S is an exact functor

<u>thm</u> $K_1(A) \cong K_0(SA)$; moreover, $\exists\, \theta_A : K_1(A) \to K_0(SA)$ isomorphisms s.t.

$$\begin{tikzcd} K_1(A) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "G"] & K_1(B) \arrow[d, "\theta_B"] \\ K_0(SA) \arrow[r, "K_0(S\varphi)"'] & K_0(SB) \end{tikzcd}$$

let $u \in \mathcal{U}_n(\tilde{A})$ with $s(u) = 1_n$, $v \in C_0([0,1], \mathcal{U}_{2n}(\tilde{A}))$ s.t. $v(0) = 1_{2n}$, $v(1) = u \oplus u^*$, $s(v(t)) = 1_{2n}$ $\forall t \in [0,1]$, let $p = v(1_n \oplus 0_n)v^*$, then $v \in P_{2n}(\widetilde{SA})$, $s(p) = 1_n \oplus 0_n$, $\theta_A([u]_1) = [p]_0 - [s(p)]_0$.

rmk note $0 \to SA \xrightarrow{\iota} CA \xrightarrow{\pi} A \to 0$ is a SES and since CA is homotopy equivalent to 0, $K_1(CA) = K_0(CA) = 0$, hence $\delta_1 : K_1(SA) \to K_0(A)$ is an isomorphism; set $\theta_A = \delta_1$.

def set $K_n(A) = K_{n-1}(SA) = K_1(S^{n-1}A)$, $K_n(\varphi) = K_{n-1}(S\varphi)$

prop K_n is half-exact

pf K_0 is half-exact and S is exact R

def higher index maps $\delta_{n+1} : K_{n+1}(B) \to K_n(I)$ defined by

$$\begin{tikzcd} K_{n+1}(B) \arrow[r, "\delta_{n+1}"] \arrow[d, equal] & K_n(I) \arrow[d, "\theta_{S^{n-1}I}"] \\ K_1(S^n B) \arrow[r, "\delta_1"'] & K_0(S^n I) \end{tikzcd}$$

prop δ_n are natural

prop a SES $0 \to I \to A \to B \to 0$ induces a long exact sequence

$$\dots \to K_{n+1}(B) \xrightarrow{\delta_{n+1}} K_n(I) \to K_n(A) \to K_n(B) \to \dots$$

ex $K_n(\mathbb{C}) \cong K_0(C_0(\mathbb{R}^n))$, $K_{n+1}(\mathbb{C}) \cong K_1(C_0(\mathbb{R}^n))$