

Introduction to the thermal bootstrap

Hidde Stoffels

March 5, 2026

1 Recap of last time

- As a refresher, what did Maegan show us two weeks ago?
- First, she reminded us that crossing symmetry poses a very powerful constraint on correlators in $T = 0$ CFT.
- We saw this by looking at a four-point function (of identical scalars, for simplicity); we can decompose it in conformal partial waves by using the OPE:

$$\left\langle \underbrace{\phi(x_1)\phi(x_2)} \underbrace{\phi(x_3)\phi(x_4)} \right\rangle = \sum_{\mathcal{O}} \frac{\lambda_{\phi\phi\mathcal{O}}^2}{c_{\mathcal{O}}^2} C_{\mathcal{O}}(x_{12}, \partial_y) C_{\mathcal{O}}(x_{34}, \partial_w) \langle \mathcal{O}(y)\mathcal{O}(w) \rangle \Big|_{\substack{y=x_2 \\ w=x_4}} \quad (1)$$

where the sum goes over all primary operators in the CFT, $\lambda_{\phi\phi\mathcal{O}}$ are OPE coefficients (i.e. $\langle \phi\phi\mathcal{O} \rangle \propto \lambda_{\phi\phi\mathcal{O}}$), $c_{\mathcal{O}}$ is the normalisation of the two-point function of \mathcal{O} with itself¹ and the $C_{\mathcal{O}}(x, \partial)$ are differential operators² that can be written as a series in $x^\mu \partial_\mu$; by applying the OPE to the three-point function $\langle \phi\phi\mathcal{O} \rangle$, you can find

$$C_{\mathcal{O}}(x, \partial_y) = \frac{1}{|x|^{2\Delta_\phi - \Delta_{\mathcal{O}}}} (1 + O(\partial_y)) \quad (2)$$

- This suggests splitting off the \mathcal{O} -independent part, defining conformal blocks $G_{\mathcal{O}}$ as

$$\left\langle \underbrace{\phi(x_1)\phi(x_2)} \underbrace{\phi(x_3)\phi(x_4)} \right\rangle = \sum_{\mathcal{O}} \frac{\lambda_{\phi\phi\mathcal{O}}^2}{c_{\mathcal{O}}^2} \frac{G_{\mathcal{O}}(u, v)}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}} \quad (3)$$

- The conformal blocks only depend on the conformally invariant cross-ratios

$$u = \frac{|x_{12}|^2 |x_{34}|^2}{|x_{13}|^2 |x_{24}|^2} \quad v = \frac{|x_{14}|^2 |x_{23}|^2}{|x_{13}|^2 |x_{24}|^2} \quad (4)$$

- You can see this is because the factor we split off already takes care of scaling and special conformal invariance, so the conformal blocks shouldn't do anything.
- However, the order in which we applied the OPE is not forced on us, we can choose a different construction by e.g. swapping $x_2 \leftrightarrow x_4$:

$$\left\langle \underbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} \right\rangle = \sum_{\mathcal{O}} \frac{\lambda_{\phi\phi\mathcal{O}}^2}{c_{\mathcal{O}}^2} \frac{G_{\mathcal{O}}(v, u)}{|x_{14}|^{2\Delta_\phi} |x_{23}|^{2\Delta_\phi}} \quad (5)$$

¹Recall that in the three-point function $\langle \phi\phi\mathcal{O} \rangle$, we can apply the OPE to find $\langle \phi\phi\mathcal{O} \rangle \sim \mathcal{C}_{\phi\phi\mathcal{O}} \langle \mathcal{O}\mathcal{O} \rangle \sim \mathcal{C}_{\phi\phi\mathcal{O}} c_{\mathcal{O}}$ for some constant $\mathcal{C}_{\phi\phi\mathcal{O}}$. However, we also know from conformal invariance that $\langle \phi\phi\mathcal{O} \rangle \sim \lambda_{\phi\phi\mathcal{O}}$, meaning that the constant which appears in the OPE is $\mathcal{C}_{\phi\phi\mathcal{O}} = \lambda_{\phi\phi\mathcal{O}}/c_{\mathcal{O}}$. This usually doesn't matter, since we'd normalise $c_{\mathcal{O}} = 1$, but for e.g. the stress tensor c_T is fixed by a Ward identity instead of convention.

²This explains why the sum only looks at primary operators; descendants are created by acting with $P^\mu = i\partial^\mu$ so the action of $C_{\mathcal{O}}$ gives you the contributions from all the descendants of a primary \mathcal{O} .

- Equating (3) and (5), we see that

$$\frac{1}{u^{\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 G_{\mathcal{O}}(u, v) = \frac{1}{v^{\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 G_{\mathcal{O}}(v, u) \quad (6)$$

- This is called crossing symmetry, because you can draw a diagram for both sides which looks like the crossing symmetry in Feynman diagrams:

$$\sum_{\mathcal{O}} \begin{array}{c} \phi(x_1) \quad \phi(x_3) \\ \diagdown \quad \diagup \\ \mathcal{O} \\ \diagup \quad \diagdown \\ \phi(x_2) \quad \phi(x_4) \end{array} = \sum_{\mathcal{O}} \begin{array}{c} \phi(x_1) \quad \phi(x_3) \\ \diagdown \quad \diagup \\ \mathcal{O} \\ \diagup \quad \diagdown \\ \phi(x_2) \quad \phi(x_4) \end{array} \quad (7)$$

where any two ‘external’ operators that are connected, are replaced by (a function times) \mathcal{O} in the OPE.

- This is non-trivial since such a symmetry is not at all manifestly obeyed by the conformal blocks (hence you need to carefully pick your OPE coefficients).
- Unfortunately, with $T \neq 0$, this is impossible: the four-point function in a thermal CFT is too complicated since the two-point function is not orthogonal. The sum in (1) actually becomes three infinite sums over all-new data³.
- Thus, we went back to the basics, and considered the one-point functions $\langle \mathcal{O}^{\mu_1 \dots \mu_J} \rangle_\beta$. Due to translation invariance, these vanish for descendants (which are created with $P^\mu = i\partial^\mu$).
- For primary operators you can, either through an application of some representation theory or by a detailed analysis of the broken Ward identities, identify the one-point functions as symmetric traceless tensors:

$$\langle \mathcal{O}^{\mu_1 \dots \mu_J} \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} (e^{\mu_1} \dots e^{\mu_J} - \text{traces}) \quad (8)$$

where e^μ is a unit vector in the (compact, and therefore distinguished) τ direction; the dependence on β follows from dimensional analysis; and the one-point coefficients $b_{\mathcal{O}}$ are the thermal data which supplements the CFT data.

- We can then consider the next simplest object: the two-point function of a neutral scalar operator with itself. We can apply the OPE with $|x|^2 = \tau^2 + |\mathbf{x}|^2 < \beta^2$:

$$g(\tau, \mathbf{x}) = \langle \phi(\tau, \mathbf{x}) \phi(0) \rangle_\beta = \sum_{\mathcal{O}} \frac{\lambda_{\phi\phi\mathcal{O}}}{c_{\mathcal{O}}} \frac{1}{|x|^{2\Delta_\phi - \Delta_{\mathcal{O}}}} \frac{x^{\mu_1}}{|x|} \dots \frac{x^{\mu_J}}{|x|} \langle \mathcal{O}^{\mu_1 \dots \mu_J}(0) \rangle_\beta \quad (9)$$

Squint to see where this comes from: $\lambda_{\phi\phi\mathcal{O}}/c_{\mathcal{O}}$ is the constant in the OPE, the x -dependence is the same as in the $T = 0$ case because we’re inside the radius of convergence; $|x|^{\Delta_{\mathcal{O}} - 2\Delta_\phi}$ comes from (2), and the $x^\mu/|x|$ contract the indices of spinning operators in the OPE without changing the scaling behaviour (they are also unit vectors from 0 to (τ, \mathbf{x}) , making them the natural candidate for this contraction).

³Ema pointed out that they are not new data in a strict sense, but for all practical intents and purposes we may as well see them as such.

- With the universal form of the one-point functions, the contractions become Gegenbauer polynomials $C_J^{(\frac{d-2}{2})}$ (orthogonal polynomials of order J), so for $\eta = \tau/|x|$ we get

$$g(\tau, \mathbf{x}) = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \frac{C_J^{(\frac{d-2}{2})}(\eta)}{|x|^{2\Delta_{\mathcal{O}} - \Delta_{\mathcal{O}}}} \quad \text{with} \quad a_{\mathcal{O}} = \frac{\lambda_{\phi\phi\mathcal{O}}}{c_{\mathcal{O}}} \frac{J! b_{\mathcal{O}}}{2^J (\frac{d-2}{2})_J} \quad (10)$$

- We can view the factor $C_J^{(\frac{d-2}{2})}(\eta)/|x|^{2\Delta_{\mathcal{O}} - \Delta_{\mathcal{O}}}$ as a two-point thermal conformal block, the thermal analogue to (3) and (5). If we want to bootstrap the thermal data, we also need an analogue to (6). The non-trivial constraint we can impose on g comes from the KMS condition, since it is not obvious from (10) that $g(\tau, \mathbf{x}) = g(\beta - \tau, \mathbf{x})$.
- As an aside, the KMS condition implies invariance⁴ under $\tau \rightarrow -\tau$ (combined with a spatial flip, as the CFT may not be parity invariant⁵). Since the Gegenbauer polynomials are only even functions of τ for even J , only even-spin operators contribute to g .
- An obstacle to using KMS as a bootstrap equation is that the $a_{\mathcal{O}}$ are not necessarily positive; this prevents us from using important bootstrap techniques⁶. We need some better tools.

2 A toy example of an inversion formula

- The tool of choice is called an ‘inversion formula’. To get an idea for what it is, we consider a toy example from Alessio Miscioscia’s thesis. Consider some unknown function $f(z)$ with $z \in \mathbb{C}$, and assume that we can write down a (formal) power series:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad (11)$$

- Next, assume we know a little bit about the analytic properties of f . For example, that it is analytic everywhere except along two branch cuts along the real axis with $|z| > 1$. Then we can use Cauchy’s theorem to extract the f_n :

$$f_n = \oint_{|z|<1} \frac{dz}{2\pi i} \frac{f(z)}{z^{n+1}} \quad (12)$$

- If we additionally assume that we know that $|f(z)/z|$ is bounded as $|z| \rightarrow \infty$, then we can go further. For $n > 1$, we can blow up the contour, wrapping it around the branch cuts and picking up discontinuities. This leads to

$$\begin{aligned} f_n &= \lim_{\epsilon \downarrow 0} \int_{-\infty}^{-1} \frac{dz}{2\pi i} \frac{f(z+i\epsilon) - f(z-i\epsilon)}{z^{n+1}} + \lim_{\epsilon \downarrow 0} \int_1^{\infty} \frac{dz}{2\pi i} \frac{f(z+i\epsilon) - f(z-i\epsilon)}{z^{n+1}} \\ &= \int_1^{\infty} \frac{dz}{2\pi z^{n+1}} (\text{Disc}[f(z)] - (-1)^n \text{Disc}[f(-z)]) \end{aligned} \quad (13)$$

Thus, we only need knowledge of the analytic structure of f to reconstruct it completely! We have expressed f_n in terms of (simple properties of) $f(z)$, instead of the other way around; therefore this is known as an inversion formula.

⁴Strictly speaking, this should be a reflection plus a shift; but since the correlator is separately invariant under time translations, a reflection is also an invariance.

⁵ g is parity invariant in the sense that we can flip any and all of the spatial coordinates, e.g. send $\mathbf{x} \rightarrow -\mathbf{x}$. However, it is not obvious that we should be able to send $x \rightarrow -x$. To account for this, we combine a flip of τ with a flip of *one* spatial coordinate, which gets flipped back if we then flip $\mathbf{x} \rightarrow -\mathbf{x}$.

⁶This is because the usual bootstrap casts (6) as a linear combination of (infinite-dimensional) vectors $v^{\Delta_{\phi}} G_{\mathcal{O}}(u, v) - u^{\Delta_{\phi}} G_{\mathcal{O}}(v, u)$, with positive coefficients $\lambda_{\phi\phi\mathcal{O}}^2$. Such a linear combination cannot vanish unless the conformal blocks satisfy certain properties, which tell you about the spectrum of the theory. This is impossible if the coefficients $a_{\mathcal{O}}$ can have any sign.

3 The Euclidean inversion formula

- Now look back at the thermal two-point function. We can rewrite it as a spectral integral:

$$g(\tau, \mathbf{x}) = \frac{1}{\beta^{2\Delta_\phi}} \sum_{J=0}^{\infty} \int_{-i\infty-\epsilon}^{i\infty-\epsilon} \frac{d\Delta}{2\pi i} a(\Delta, J) C_J^{(\frac{d-2}{2})}(\eta) \frac{1}{(|x|/\beta)^{2\Delta_\phi-\Delta}} \quad (14)$$

where we continued the thermal coefficient $a_{\mathcal{O}}$ to a function $a(\Delta, J)$ with simple poles that look like

$$a(\Delta, J) \sim -\frac{a_{\mathcal{O}}}{\Delta - \Delta_{\mathcal{O}}} \quad (15)$$

We also assume that $a(\Delta, J)$ does not grow exponentially in the right-half plane. Then we can close the contour through this half plane to pick up the residues at these poles (the fact that we're going clockwise also explains the minus sign).

- What (14) does is that it replaces a sum over all primaries with an integral over all possible (complex) scaling dimensions a primary operator of spin J could have (picking up only contributions from the operators actually present in the theory) and then summing over all possible values of J .
- To get an inversion formula, we would like to isolate $a(\Delta, J)$. First, we pick out one term in the sum over the spins, using the orthogonality of the Gegenbauer polynomials:

$$\int_{S^{d-1}} d\Omega_{d-1} C_J^{(\frac{d-2}{2})}(\eta) C_{J'}^{(\frac{d-2}{2})}(\eta) = N_J \delta_{J'J} \quad (16)$$

with the normalisation constant⁷

$$N_J = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \frac{2^{1-(d-2)} \pi \Gamma(J+d-2)}{(J+\frac{d-2}{2}) \Gamma^2(\frac{d-2}{2}) \Gamma(J+1)} \quad (17)$$

- Integrating g implies that

$$\int_{-i\infty-\epsilon}^{i\infty-\epsilon} \frac{d\Delta}{2\pi i} a(\Delta, J) \left(\frac{|x|}{\beta}\right)^\Delta = \frac{\beta^{2\Delta_\phi-d}}{N_J} \int_{S^{d-1}} |x|^d d\Omega_{d-1} C_J^{(\frac{d-2}{2})}(\eta) g(\tau, \mathbf{x}) (|x|/\beta)^{2\Delta_\phi-d} \quad (18)$$

Now write the object on the left in a weird way:

$$\int_{-i\infty-\epsilon}^{i\infty-\epsilon} \frac{d\Delta}{2\pi i} a(\Delta, J) e^{\ln(|x|/\beta)\Delta} = \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{d\Delta'}{2\pi i} a'(\Delta', J) e^{s\Delta'} = A(s) \quad (19)$$

with $\Delta' = -\Delta$, $a'(\Delta', J) = a(\Delta, J)$, and $s = -\ln(|x|/\beta)$.

- If you are familiar with complex analysis, then you can see that $A(s)$ is the inverse Laplace transform of $a'(\Delta', J)$ so you can take the (usual) Laplace transform to extract $a(\Delta, J)$.
- Let's show this. We write down the Laplace transform of $A(s)$:

$$\mathcal{L}[A](\omega) = \int_0^\infty e^{-s\omega} A(s) ds = \int_0^\infty ds \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{d\Delta'}{2\pi i} a'(\Delta', J) e^{-s(\omega-\Delta')} \quad (20)$$

⁷This might look somewhat different from the normalisation one finds on Wikipedia. The difference is that we are integrating over an S^{d-1} ; the integral over the polar angle, here given by $\cos\theta = \tau/|x|$, produces the normalisation from Wikipedia, while the integrand does not depend on the integral over the remaining S^{d-2} , which therefore just produces a prefactor $2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2})$. Also note that $N_J = 0$ for $d = 2$, this case is treated separately.

- Now we assume that $\text{Re}(\omega) > \epsilon$ so we can evaluate the s -integral:

$$\mathcal{L}[A](\omega) = \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{d\Delta'}{2\pi i} \frac{a'(\Delta', J)}{\omega - \Delta'} \quad (21)$$

- The poles of a' are all in the left half of the complex Δ' plane (because poles occur in the right half of the Δ plane), but the pole at $\Delta' = \omega$ is in the right half (since $\text{Re}(\omega) > \epsilon > 0$). If we assume that a' decays sufficiently fast, we can close the contour either way; since the outcome shouldn't depend on which way we go, we choose to go through the right-hand side and pick up the pole at ω :

$$\mathcal{L}[A](\omega) = a'(\omega, J) = a(-\omega, J) \quad (22)$$

- Supposedly, this only works for $\omega > 0$, so if we want $a(\Delta, J) = \mathcal{L}[A](-\Delta)$ with $\Delta > 0$ we are not going to do well. However, this should not matter: $A(s)$ has no poles as a function of s , so we can freely deform the contour in $\mathcal{L}[A](-\Delta)$ to run (almost) along the negative real axis, absorbing the minus sign from $-\Delta$.
- Therefore, it follows that

$$\begin{aligned} a(\Delta, J) &= \int_0^\beta \frac{d|x|}{|x|} (|x|/\beta)^{-\Delta} \frac{\beta^{2\Delta_\phi-d}}{N_J} \int_{S^{d-1}} |x|^d d\Omega_{d-1} C_J^{(\frac{d-2}{2})}(\eta) g(\tau, \mathbf{x}) (|x|/\beta)^{2\Delta_\phi-d} \\ &= \frac{\beta^{2\Delta_\phi-d}}{N_J} \int_{|x|<\beta} d^d x C_J^{(\frac{d-2}{2})}(\eta) g(\tau, \mathbf{x}) (|x|/\beta)^{2\Delta_\phi-\Delta-d} \end{aligned} \quad (23)$$

- This expresses the thermal one-point data in terms of an integral of the two-point function over a Euclidean sphere (such that the OPE converges everywhere inside the sphere). This might seem daft: why compute the one-point coefficients in terms of the two-point function? You'd want to go the other way!
- The justification becomes apparent once we go to a Lorentzian spacetime, and express $a(\Delta, J)$ in terms of discontinuities of g . Then knowledge of only the analytic structure of the two-point function is required, which is easier to obtain.

4 Continuation to Lorentzian signature

- We arrived at the (Euclidean) thermal manifold by Wick rotating Lorentzian time into the Euclidean time τ . So we would expect to return to a Lorentzian manifold by Wick rotating τ again.
- This is not what happens in the thermal bootstrap. Instead, we Wick rotate one of the spatial coordinates. The reasoning is somewhat a posteriori; a priori, we can see that it is quite convenient that the new Lorentzian time is automatically non-compact.
- For now, we note that the Euclidean inversion formula (23) integrates over a region in (real) Euclidean space; but in our toy example of an inversion formula (23) we used properties of f in the complex plane.
- To reformulate (23) as a contour integral in \mathbb{C} , let $r = |\mathbf{x}|$ be the radial (spatial) coordinate in Euclidean signature, and then define complex coordinates

$$z = \tau + ir \quad \bar{z} = \tau - ir \quad (24)$$

Also define a polar angle θ (in the (τ, r) -plane) and $w = e^{i\theta}$ such that

$$z = |x|w \quad \bar{z} = \frac{|x|}{w} \quad (25)$$

- Let's first write the Euclidean correlator in terms of $|x|$ and w . Observe that the integrand of (23) only depends on $r = |\mathbf{x}|$ due to the $\text{SO}(d-1)$ invariance; therefore we can quickly evaluate the angular spatial integrals, leading to

$$\begin{aligned}
a(\Delta, J) &= \frac{\beta^\Delta}{N_J} \int_{\tau^2+r^2<\beta^2} d\tau dr \frac{2\pi^{\frac{d-1}{2}} r^{d-2}}{\Gamma(\frac{d-1}{2})} C_J^{(\frac{d-2}{2})} \left(\frac{\tau}{\sqrt{\tau^2+r^2}} \right) g(\tau, r) (\tau^2+r^2)^{\Delta_\phi - \frac{\Delta+d}{2}} \\
&= \frac{(J + \frac{d-2}{2})\Gamma^2(\frac{d-2}{2})\Gamma(J+1)}{2^{3-d}\pi\Gamma(J+d-2)} \int_0^\beta d|x| |x|^{2\Delta_\phi - \Delta - d + 1} \oint \frac{dw}{iw} C_J^{(\frac{d-2}{2})} \left(\frac{w+w^{-1}}{2} \right) \\
&\quad \times g(z, \bar{z}) \left(\frac{z-\bar{z}}{2i} \right)^{d-2}
\end{aligned} \tag{26}$$

where the w -integral is over a unit circle (it is precisely the θ integral) and z and \bar{z} should be viewed as functions of $|x|$ and w .

- Next we consider the behaviour of the integrand as a function of w . For one, the Gegenbauer polynomial can be rewritten to

$$C_J^{(\frac{d-2}{2})} \left(\frac{w+w^{-1}}{2} \right) = \frac{i^{d-2}\Gamma(J+d-2)}{\Gamma(\frac{d-2}{2})\Gamma(J+\frac{d}{2})} \left(F_J(w^{-1}) + (-1)^{d-2} F_J(w) \right) \tag{27}$$

with the F_J solutions of the Gegenbauer differential equation:

$$F_J(w) = w^{J+d-2} {}_2F_1 \left(J+d-2, \frac{d}{2}-1, J+\frac{d}{2}, w^2 \right) \tag{28}$$

Note that $F_J(w) \rightarrow 0$ for $w \rightarrow 0$.

- Now, we make two assumptions about the analytic properties of g :
 - g has branch cuts along the real axis in the intervals $(-\infty, -\beta/|x|)$, $(-|x|/\beta, 0)$, $(0, |x|/\beta)$ ⁸, and $(\beta/|x|, \infty)$, but is analytic in the rest of the w plane;
 - And g is limited by w^{J_*} for some J_* as $|w| \rightarrow \infty$, and by w^{-J_*} as $|w| \rightarrow 0$.
- We justify the first in a moment; the second hasn't yet been justified anywhere beyond 'it just works'. For now, we can deform the w contour for the $F_J(w)$ and $F_J(w^{-1})$ term.
- Consider first the latter. It dies off as $|w| \rightarrow \infty$, so we blow up the unit circle, instead only getting contributions from the branch cuts that extend to infinity⁹.
- The branch cut on the right contributes an integral from $w = \beta/|x|$ to infinity of the discontinuity of g across the cut (the factor $(z-\bar{z})^{d-2}$ has no branch cuts, and neither does F_J).
- The branch cut on the left actually contributes almost the same: due to the invariance of g under a combined τ and r flip, we have $g(z, \bar{z}) = g(-z, -\bar{z})$, meaning that the two-point function is invariant under $w \rightarrow -w$. Then, F_J produces a factor $(-1)^{J+d-2}$, $(z-\bar{z})^{d-2}$ a factor $(-1)^{d-2}$, and the explicit factor $1/w$ produces a single minus sign; finally, the discontinuity is defined as the limit of the function from above the real axis minus the limit from below the real axis; thus, flipping $w \rightarrow -w$ introduces an additional minus sign. All in all, the branch cut that stretches to $-\infty$ contributes the same as the branch cut which stretches to $+\infty$ but with a factor $(-1)^J$.

⁸There is a pole between these two intervals.

⁹In principle, there could be a contribution from the arc at infinity as well if $J \leq J_*$ because the Gegenbauer term (F_J) then doesn't necessarily decay sufficiently fast to kill off the two-point function. We ignore these for simplicity (these terms are correlator-dependent).

- For the $F_J(w)$ term, we deform the contour to wrap around the branch cut at $(-|x|/\beta, |x|/\beta)$. Then observe that the two halves of this branch cut can be mapped to the branch cuts reaching infinity by sending $w \rightarrow w^{-1}$; this transformation sends $F_J(w) \rightarrow F_J(w^{-1})$ and leaves the two-point function invariant¹⁰; and finally it sends $(z - \bar{z})^{d-2} \rightarrow (-1)^{d-2}(z - \bar{z})^{d-2}$. The prefactor exactly compensates the factor $(-1)^{d-2}$ which appears in the Gegenbauer decomposition, so the contribution from $F_J(w)$ is exactly equal to the contribution from $F_J(w^{-1})$.
- Putting everything together, we find that

$$a(\Delta, J) = \frac{(1 + (-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{\pi\Gamma(J + \frac{d-2}{2})} \int_0^\beta d|x| |x|^{2\Delta_\phi - \Delta - d + 1} \int_{\beta/|x|}^\infty \frac{dw}{w} F_J(w^{-1}) \times \text{Disc}[g(z, \bar{z})](z - \bar{z})^{d-2} \quad (29)$$

Observe that it vanishes whenever J is odd, in accordance with our expectations about which operators have non-vanishing one-point functions.

- Now switch coordinates to z, \bar{z} . The measure is $|x|d|x|\frac{dw}{w} = \frac{1}{2}dzd\bar{z}$; we also know that $z = |x|w \geq \beta$, and because $|x|^2 = z\bar{z} \leq \beta^2$ we get that z runs from β to β^2/\bar{z} while \bar{z} goes from $\bar{z} = 0$ (when $w \rightarrow \infty$) up to $\bar{z} = \beta$ as $w \rightarrow \beta^2/|x|$ and $|x| \rightarrow \beta$. We finally find that

$$a(\Delta, J) = \frac{(1 + (-1)^J)\Gamma(\frac{d-2}{2})\Gamma(J+1)}{2\pi\Gamma(J + \frac{d-2}{2})} \int_0^\beta d\bar{z} \int_\beta^{\beta/\bar{z}} dz (z\bar{z})^{\Delta_\phi - \frac{\Delta+d}{2}} (z - \bar{z})^{d-2} F_J\left(\sqrt{\bar{z}/z}\right) \times \text{Disc}[g(z, \bar{z})] \quad (30)$$

The discontinuity is to be taken by fixing $z\bar{z} = |x|^2$ and then allowing the coordinate in the direction orthogonal to this curve (i.e. $\sqrt{z/\bar{z}}$) to take on imaginary values.

- This is called a Lorentzian inversion formula, because z and \bar{z} are now independent real coordinates; we can interpret them as lightcone coordinates on a cylinder with Lorentzian time $x_L = ir$ (instead of the usual $t = i\tau$). Also, we only need to know about discontinuities in g .

5 Analyticity in the complex w plane

- To finish the proof we need to justify our assumptions about the analyticity of g . As mentioned, the bound on g as w goes to 0 or ∞ has no known justification beyond there being no known counterexamples.
- For the analyticity in most of the plane, we can view the thermal manifold as a Wick rotation of the Lorentzian manifold $\mathbb{R} \times (S_\beta^1 \times \mathbb{R}^{d-2})$ which is a Kaluza-Klein compactification of Minkowski spacetime.
- Therefore, we can write the two-point function as

$$\begin{aligned} g(\tau, \mathbf{x}) &= \langle 0 | \phi(0) e^{-H_{\text{KK}}r + iP_{\text{KK}}\tau} \phi(0) | 0 \rangle \\ &= \langle 0 | \phi(0) e^{\frac{i}{2}(H_{\text{KK}} + P_{\text{KK}})z - \frac{i}{2}(H_{\text{KK}} - P_{\text{KK}})\bar{z}} \phi(0) | 0 \rangle \end{aligned} \quad (31)$$

where H_{KK} generates translations in Lorentzian time x_L and P_{KK} generates translations in the spatial direction τ .

¹⁰I think this is because this change is flipping the spatial position (it reverses only the sign of r).

- Now observe that we can write

$$e^{\frac{i}{2}(H_{\text{KK}}+P_{\text{KK}})z-\frac{i}{2}(H_{\text{KK}}-P_{\text{KK}})\bar{z}} = V^{\frac{1}{2}}UV^{\frac{1}{2}} \quad (32)$$

with U and V defined as

$$U = e^{\frac{i}{2}(H_{\text{KK}}+P_{\text{KK}})\text{Re}(z)-\frac{i}{2}(H_{\text{KK}}-P_{\text{KK}})\text{Re}(\bar{z})} \quad (33)$$

$$V = e^{-\frac{1}{2}(H_{\text{KK}}+P_{\text{KK}})\text{Im}(z)+\frac{1}{2}(H_{\text{KK}}-P_{\text{KK}})\text{Im}(\bar{z})} \quad (34)$$

This is very straightforward since H_{KK} and P_{KK} commute. Note that $V^\dagger = V$ and $U^\dagger = U^{-1}$.

- If we now define $|A\rangle = V^{\frac{1}{2}}\phi^\dagger(0)|0\rangle$ and $|B\rangle = UV^{\frac{1}{2}}\phi(0)|0\rangle$, then we can use the Cauchy-Schwarz inequality to see that

$$|g(\tau, \mathbf{x})| = |\langle A|B\rangle| \leq \sqrt{\langle A|A\rangle\langle B|B\rangle} = |\langle 0|\phi(0)V\phi(0)|0\rangle| \quad (35)$$

- Then, let's assume that the spectrum of $H_{\text{KK}} \pm P_{\text{KK}}$ is bounded below by λ . This is not rigorously proven¹¹, but it seems reasonable; after all, for a very big momentum in the compact direction, you probe small distances and should be insensitive to the compactness. If you then decrease this momentum by a finite amount (to go to the low-momentum limit), the eigenvalues of $H_{\text{KK}} \pm P_{\text{KK}}$ should not suddenly diverge.
- We can also define $\zeta = \min(\text{Im}(z), -\text{Im}(\bar{z}))$ so that $\text{Im}(z) - \zeta > 0$ and $-\text{Im}(\bar{z}) - \zeta > 0$. It then follows that

$$\begin{aligned} |g(\tau, \mathbf{x})| &\leq |\langle 0|\phi(0)e^{-\frac{1}{2}(H_{\text{KK}}+P_{\text{KK}})\text{Im}(z)+\frac{1}{2}(H_{\text{KK}}-P_{\text{KK}})\text{Im}(\bar{z})} \\ &\quad \times e^{\frac{1}{2}(H_{\text{KK}}+P_{\text{KK}}-\lambda)(\text{Im}(z)-\zeta)}e^{\frac{1}{2}(H_{\text{KK}}-P_{\text{KK}}-\lambda)(-\text{Im}(\bar{z})-\zeta)}\phi(0)|0\rangle| \\ &= |\langle 0|\phi(0)e^{-\frac{1}{2}(H_{\text{KK}}+P_{\text{KK}})\zeta-\frac{1}{2}(H_{\text{KK}}-P_{\text{KK}})\zeta}\phi(0)|0\rangle|e^{-\frac{\lambda}{2}(\text{Im}(z)-\zeta)}e^{\frac{\lambda}{2}(\text{Im}(\bar{z})+\zeta)} \\ &= |g(0, r = \zeta)|e^{-\frac{\lambda}{2}(\text{Im}(z)-\zeta)}e^{\frac{\lambda}{2}(\text{Im}(\bar{z})+\zeta)} \end{aligned} \quad (36)$$

- Clearly, this expression makes sense (and gives an upper bound to the correlator) if $\zeta > 0$, i.e. if $\text{Im}(z) > 0$ and $\text{Im}(\bar{z}) < 0$; by (25), this means that w is in the upper half of the complex plane.
- We see that $g(0, r = \zeta)$ is finite (it is a correlator where the operators are inserted a finite distance away from each other) and that the remaining exponential factor is suppressing it; therefore, $g(\tau, \mathbf{x})$ is finite when w is in the upper half of \mathbb{C} . So g is analytic there.
- This conclusion can be extended to the lower half of \mathbb{C} by using the symmetry under $w \rightarrow -w$. All branch cuts must therefore be on the real axis in the w plane.

6 Further reading

- [EPFL Lectures on Conformal Field Theory in \$d \geq 3\$ Dimensions](#) and [TASI Lectures on the Conformal Bootstrap](#) expand on the OPE, conformal blocks, and the $T = 0$ conformal bootstrap;
- [The Conformal Bootstrap at Finite Temperature](#) originally proposed the thermal inversion formula;
- [Thermal effects in conformal field theories](#) gives a slightly more pedagogical introduction to the topic.

¹¹The problem is that the eigenvalues of P_{KK} are discrete, so there is no Lorentz transformation relating $H_{\text{KK}} \pm P_{\text{KK}}$ to the Hamiltonian of some timelike observer. Hence positivity of energy doesn't seem to work.