Introduction and summary

C_T : central charge d-dim CFT at large C_T

$\mathcal{O}_H$: heavy operator $\mathcal{O}_L$: light operator

$\Delta_H \sim C_T$

multi stress tensor $T^k_{\tau,s}$: k stress tensors with twist τ and spin s

OPE coeff scale like | $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^k_{\tau,s}} \sim \dfrac{\Delta^k}{C_T^{k/2}}$ for large Δ

The contribution of a given multi stress tensor operator to HHLL : $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ can be compared to the contribution of the same operator to $\langle \mathcal{O}_L \mathcal{O}_L \rangle_\beta$

$\langle \mathcal{O}_H T^k_{\tau,s} \mathcal{O}_H \rangle = \langle T^k_{\tau,s} \rangle_\beta$ β is related to Δ_H

$\hookrightarrow$ Thermalization of the stress tensor sector (related to ETH)

Thermalization and universality

large $-C_T$ CFTs on $S^{d-1}\times\mathbb{R}$ with radius R

We consider HHLL correlators

$2\ \mathcal{O}_H$ at $x_E^0=\pm\infty$

conformal tranf to the plane

$$\langle \mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle = \lim_{x_4\to\infty} x_4^{2\Delta_H}(z\bar{z})^{-\Delta/2}\langle \mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle$$

cross-ratios $(z,\bar{z})$: $\quad z=e^{-x_E^0-i\varphi} \qquad \bar{z}=e^{-x_E^0+i\varphi}$

The stress tensor sector of HHLL:

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H}\langle \mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi stress tensor}} =$$

expanded in conformal blocks

$$= \frac{1}{[(1-z)(1-\bar{z})]^{\Delta}}\sum_{T^k_{\tau,s}} P^{(HHLL)}_{T^k_{\tau,s}}\, g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})$$

conformal blocks

τ, s, k label the twist, spin and multiplicity of multi stress tensor (the twist is defined by $\tau = \Delta - s$)

We are interested in the double scaling limit:

$$C_T, \Delta_H \longrightarrow \infty \quad , \mu \propto \frac{\Delta_H}{C_T} \quad \text{fixed}$$

In this limit:

$$P^{(HHLL)}_{T^k_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}} \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}} \Big|_{\left(\frac{\Delta_H}{\sqrt{C_T}}\right)^k}$$

only keep the leading $\left(\frac{\Delta_H}{\sqrt{C_T}}\right)^k$ term in the OPE coeff

Contribution of the conformal family of a particular $T^k_{\tau,s}$:

$$\langle \mathcal{O}_H | \mathcal{O}(x_E^0, \varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle\Big|_{T^k_{\tau,s}} = \frac{P^{(HHLL)}_{T^k_{\tau,s}} \; g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z})}{\left[\sqrt{z\bar{z}}\,(1-z)(1-\bar{z})\right]^{\Delta}}$$

CFT at finite β^{-1}:

$$\langle \mathcal{O}(x_E^\circ, \varphi)\,\mathcal{O}(0)\rangle_\beta = \frac{1}{Z(\beta)} \sum_i e^{-\beta \Delta_i} \langle \mathcal{O}_i | \mathcal{O}(x_E^\circ, \varphi)\,\mathcal{O}(0) | \mathcal{O}_i \rangle =$$

$$= \frac{1}{\left[\sqrt{z\bar{z}}\,(1-z)(1-\bar{z})\right]^{\Delta}} \sum_{T^k_{\tau,s}} \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}\, g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z})\, \langle T^k_{\tau,s}\rangle_\beta + \cdots$$

contributions from other operators

where $$\langle T^k_{\tau,s}\rangle_\beta = \frac{1}{Z(\beta)} \sum_i e^{-\beta\Delta_i} \lambda_{\mathcal{O}_i \mathcal{O}_i T^k_{\tau,s}}$$

sum over all operators, including descendants

$$\langle T^k_{\tau,s}\rangle_\beta = \beta^{-(\tau+s)} f^k_{\tau,s}(\beta) \qquad f^k_{\tau,s}(\beta) \sim C_T^{k/2}$$

theory - dependent

Thermalization of the stress tensor sector:

$$\langle \mathcal{O}_H | T^k_{\tau,s} | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T^k_{\tau,s}\rangle_\beta$$

To determine β we can consider $T^{\mu\nu}$

$f^1_{d-2,2}(\beta)$ is determined by the free energy on the sphere

Let's consider $T^k_{\tau,s}$ with no derivatives

(the conclusions do not change in the general case)

Assuming large-C_T factorization:

$$\langle T^k_{\tau,s} \rangle_\beta = c^k_{\tau,s} \left(\langle T^1_{d-2,2} \rangle_\beta \right)^k + \dots$$

nummerical coeff (depend only on k, τ, s)

subleading in C_T^{-1}

$$\lambda_{O_H O_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = C^k_{\tau,s} \left(\lambda_{O_H O_H T^1_{\tau,s}} \right)^k = C^k_{\tau,s} \left(\frac{d}{1-d} \right)^k \frac{\Delta_H^k}{C_T^{k/2}}$$

Ward identity for the three point function

OPE coefficients in the free adjoint scalar model

4d, free scalar in the adjoint rep of SU(N)

$$C_T = \frac{4}{3}(N^2-1)$$

$$\langle \phi^i{}_j(x)\, \phi^k{}_l(y) \rangle = \left(\delta^i{}_l \delta^k{}_j - \frac{1}{N} \delta^i{}_j \delta^k{}_l \right) \frac{1}{|x-y|^2}$$

Single trace scalar operator with dim Δ;

$$\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta}\, N^{\frac{\Delta}{2}}} :\mathrm{Tr}(\phi^\Delta): (x)$$

$$\langle \mathcal{O}_\Delta(x)\, \mathcal{O}_\Delta(y) \rangle = \frac{1}{|x-y|^{2\Delta}}$$

Stress tensor

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} :\mathrm{Tr}\left(\partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2} \phi\, \partial_\mu \partial_\nu \phi - (\text{trace}) \right): (x)$$

$$\langle T^{\mu\nu}(x)\, T_{\rho\sigma}(0) \rangle = \frac{1}{|x|^8} \left(I^{(\mu}{}_\rho(x)\, I^{\nu)}{}_\sigma(x) - (\text{traces}) \right)$$

$$I^\mu{}_\nu(x) = \delta^\mu{}_\nu - \frac{2 x^\mu x_\nu}{|x|^2}$$

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^1_{2,2}} = -\frac{4\Delta}{3\sqrt{C_T}}$$

Thermal one-point functions in the free adjoint scalar model

$$\langle \mathcal{O}_{\mu_1 \cdots \mu_{s_0}} \rangle_\beta = \frac{b_0}{\beta^{\Delta_0}} \left(e_{\mu_1} \cdots e_{\mu_{s_0}} - \text{traces} \right)$$

e_μ: unit vector along the thermal circle

Heavy operator: $\mathcal{O}_H$ $\quad \Delta_H \sim N^2$

Stefan-Boltzmann's law $\quad \dfrac{E}{\mathrm{vol}(S^3)} = \dfrac{N^2 \pi^2}{30 \beta^4}$

the energy of a state is given by:

$$E = \frac{\Delta}{R}$$

$$\mu = \frac{160}{3} \frac{\Delta_H}{C_T} = \frac{160}{3} E \frac{R}{C_T} \approx \frac{8}{3} \left(\frac{\pi R}{\beta} \right)^4$$

Stress tensor

$$T^1_{2,2} = T_{\mu\nu}$$

It can be shown that: (Appendix D)

$$b_{T^1_{2,2}} = -\frac{2\pi^4 N}{15\sqrt{3}}$$

using

$$\lambda_{O_H O_H T^1_{2,2}} = -\frac{4\Delta_H}{3\sqrt{C_T}}$$

and

$$20\,\frac{\Delta_H}{C_T} = \left(\frac{\pi R}{\beta}\right)^4$$

$$\Rightarrow \quad b_{T^1_{2,2}}\,\beta^{-4} = \lambda_{O_H O_H T^1_{2,2}}$$

Comparison with the Eigenstate Thermalization Hypothesis

The stress tensor sector of free $SU(N)$ adjoint scalar in $d=4$ satisfies ETH to leading order in $C_T \sim N^2 \gg 1$

Equivalence between micro-canonical and canonical ensemble on $S^1_\beta \times S^{d-1}$ when $\Delta_H \sim C_T \gg 1$

$E = \frac{\Delta_H}{R}$:
$$\langle O \rangle_E^{(micro)} = \frac{1}{N(E)} \sum_{\tilde{O}} \langle \tilde{O} | O | \tilde{O} \rangle$$

$N(E)$ states $|\tilde{O}\rangle$ with energy $(E, E+\delta E)$

partition function:

$$Z(\beta) = \sum_{\tilde{O}} e^{-\frac{\beta \tilde{\Delta}}{R}} = \int d\tilde{\Delta} \ \rho(\tilde{\Delta}) e^{-\frac{\beta \tilde{\Delta}}{R}}$$

sum over all state in the theory

$$\langle O \rangle_\beta = Z(\beta)^{-1} \int d\tilde{\Delta} \cdot \rho(\tilde{\Delta}) \ \langle O \rangle_E^{micro} \ e^{-\frac{\beta \tilde{\Delta}}{R}}$$

free energy $F = -\beta^{-1} \log Z(\beta)$

$$\rho(\Delta_H) = \frac{1}{2\pi i R} \int d\beta' \, e^{\beta'\left(\frac{\Delta_H}{R} - F(\beta')\right)}$$

For $\Delta_H \sim C_T$ and $F \sim C_T$ we can use the saddlepoint approx, with saddle at β:

$$\frac{\Delta_H}{R} = \partial_{\beta'} (\beta' F)\Big|_{\beta}$$

$$\rho(\Delta_H) \langle O \rangle^{(\text{micro})}_{\Delta_H/R} = \frac{1}{2\pi i R} \int d\beta' \, \langle O \rangle_{\beta'} \, e^{\beta'\left(\frac{\Delta_H}{R} - F(\beta')\right)}$$

For $F \sim C_T \gg 1$ saddle point

$$\langle O \rangle^{(\text{micro})}_{\Delta_H/R} \simeq \langle O \rangle_{\beta}$$ with β:

$$\frac{\Delta_H}{R} = \partial_{\beta'} (\beta' F)\Big|_{\beta}$$

In the limit $R \to \infty$

$$F = \frac{b_{T_{\mu\nu}}^{(\text{can})} S_d R^{d-1}}{d \cdot \beta^d}$$

$$S_d = \mathrm{Vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

$$\frac{\beta}{R} = \left(\frac{(1-d)\, b_{T_{\mu\nu}^{(can)}}\, S_d}{d\cdot\Delta_H}\right)^{\frac{1}{d}}$$

$$\langle T_{00}^{(can)} \rangle_\beta = \frac{1}{S_d R^{d-1}}\, \partial_\beta\left(-\beta F(\beta)\right)$$

$$\langle O_H | T_{00}^{(can)} | O_H \rangle = -\frac{\Delta_H}{S_d R^d} \qquad \left(\text{fixed by Ward identity}\right)$$

$$\langle O_H | T_{00}^{(can)} | O_H \rangle = \langle T_{00}^{(can)} \rangle_\beta$$

ETH: $$\langle O_H | O_{T,S} | O_H \rangle = \langle O_{T,S} \rangle_E^{(micro)} + \mathcal{O}\left(e^{-S(E)}\right)$$

local primary operators

$$E = \frac{\Delta_H}{R}$$

$$\Delta_H \propto C_T \gg 1$$

$$S_H = 0$$

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H \mathcal{O}_{\tau,s}}}{R^{\tau+s}} = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\beta/R)}{\beta^{\tau+s}} + \mathcal{O}\left(e^{-S(E)}\right)$$

$\mathcal{O}_2 = \frac{1}{\sqrt{2}N} :\mathrm{Tr}(\phi^2):$ does not satisfy ETH