Background on Kan extensions & adjunctions

Def.: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ such that there is a natural isomorphism of functors $\mathcal{C}^{op}\times\mathcal{D}\to \mathrm{Set}$, $\varphi:\mathcal{D}(\underbrace{F}_{\text{left adjoint}}(-),-)\to\mathcal{C}(-,\underbrace{G}_{\text{right adjoint}}(-))$.

$F\dashv G$

Def.: Let $F:\mathcal{C}\to\mathcal{E}$ and $K:\mathcal{C}\to\mathcal{D}$ be functors. The left Kan extension of F along K is the functor $\mathrm{Lan}_K F:\mathcal{E}\to\mathcal{D}$ together with a natural transformation $\eta: F\Rightarrow \mathrm{Lan}_K F\circ K$ such that for any functor $G:\mathcal{D}\to\mathcal{E}$ and natural transformation $\varepsilon: F\Rightarrow G\circ K$ there is a unique natural transformation $\mu:\mathrm{Lan}_K F\Rightarrow G$ with $\varepsilon=\mu\circ\eta$.

Warning: Even if $F=H\circ K$ we do not have $\mathrm{Lan}_K F = H$.

Theorem: If $\mathcal{C}$ is small, $\mathcal{D}$ is locally small, and $\mathcal{E}$ is cocomplete, then the left Kan extension of any functor $F:\mathcal{C}\to\mathcal{E}$ along any functor $K:\mathcal{C}\to\mathcal{D}$ is given by $\mathrm{Lan}_K F(d)\cong \mathrm{colim}(K/d\to\mathcal{C}\xrightarrow{F}\mathcal{E})$ where K/d is the category with objects pairs $(c\in\mathcal{C}, Kc\xrightarrow{\phi} d)$ and morphisms $(c,\phi)\to(c',\phi')$ morphisms $\alpha: c\to c'$ in $\mathcal{C}$ such that

$$\begin{tikzcd} Kc \arrow[rr, "K\alpha"] \arrow[dr, "\phi"'] & & Kc' \arrow[dl, "\phi'"] \\ & d & \end{tikzcd}$$

commutes (this is called the comma category) and the functor $K/d\to\mathcal{C}$ is the forgetful functor $(c,\phi)\mapsto c$.

§8. Simplicial methods in homological algebra

Def.: Let Δ be the category of finite ordered sets with morphisms nondecreasing monotone functions. The objects of Δ are of the form $[n] = (0 \leq 1 \leq \dots \leq n)$.

Let $\mathcal{A}$ be a category, a simplicial object in $\mathcal{A}$ is a functor $\Delta^{op} \longrightarrow \mathcal{A}$. We denote $s\mathcal{A}$ the category of simplicial objects in $\mathcal{A}$ with morphisms natural transformations.

Examples:

① $\Delta^n \in sSet$ given as the representable functor $\mathrm{Hom}_\Delta(-, [n])$.

② $\mathrm{Sing}_\bullet(X) \in sSet$ for some topological space X is defined as $\mathrm{Sing}_n(X) = \mathrm{Sing}_\bullet(X)([n]) = \{\sigma : \mathbb{\Delta}^n \longrightarrow X\} = \mathrm{Hom}_{Top}(\mathbb{\Delta}^n, X)$ where $\mathbb{\Delta}^n = \{t \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=0}^n t_i = 1\}$.

We get a functor
$$\begin{array}{ccc} \Delta & \xrightarrow{\mathbb{\Delta}^\bullet} & Top \\ {[n]} & \longmapsto & \mathbb{\Delta}^n \end{array}$$
furthermore we may consider the Yoneda embedding
$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^\bullet} & sSet \\ {[n]} & \longmapsto & \Delta^n \end{array}.$$
Since Top is complete and cocomplete we have
$$\begin{tikzcd} \Delta \arrow[r, "\mathbb{\Delta}^\bullet"] \arrow[d, "\Delta^\bullet"'] & Top \\ sSet \arrow[ur, dashed, "|-|"'] & \end{tikzcd}$$
its left Kan-extension.

This is given by the formula $|X| = \mathrm{colim}\,(\Delta^\bullet/X \longrightarrow \Delta \xrightarrow{\mathbb{\Delta}^\bullet} Top)$

where $\Delta^\cdot/X$ is the category with objects pairs $([n]\in\Delta, \Delta^n \to X)$ and morphisms $([n],\phi) \to ([m],\psi)$ are given as maps $[n] \xrightarrow{\alpha} [m]$ s.t.

$$\begin{tikzcd} \Delta^n \arrow[rr, "\Delta(\alpha)"] \arrow[dr, "\phi"'] & \arrow[d, phantom, "\circlearrowright"] & \Delta^m \arrow[dl, "\psi"] \\ & X & \end{tikzcd}.$$

The functor $\Delta^\cdot/X \longrightarrow \Delta$ is the forgetful one.

$$([n],\phi) \longmapsto \Delta^n$$

So the colimit can be described as

$$|X| \cong \Big(\coprod_{[n]\in\Delta} X_n \times \Delta^n \Big) / \sim$$

where $(\theta^*(x), t) \sim (x, \theta_*(t))$ for all $t \in \Delta^n$, $x \in X_m$

$\theta : [n] \to [m]$.

<u>Remark</u>: $([n], \phi : \Delta^n \to X)$ corresponds to some element in X_n via Yoneda, so in $\Delta^\cdot/X \to \Delta \to \mathrm{Top}$ each element of X_n is mapped to Δ^n.

<u>Proposition</u>: $|\cdot| \dashv \mathrm{Sing}_\cdot$

<u>Proof</u>: $\mathrm{Hom}_{\mathrm{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathrm{Top}}(\mathrm{colim}(\Delta^\cdot/X \to \Delta \xrightarrow{\Delta^\cdot} \mathrm{Top}), Y)$

$$\cong \lim_{([n],\phi)\in\Delta^\cdot/X} \big(\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, Y)\big) \overset{(\star)}{\cong} \lim_{([n],\phi)\in\Delta^\cdot/X} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{Sing}_\cdot(Y))$$

$$\cong \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{colim}(\Delta^\cdot/X \to \Delta \xrightarrow{\Delta^\cdot} \mathrm{sSet}), \mathrm{Sing}_\cdot(Y))$$

$$\overset{(✡)}{\cong} \mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{Sing}_\cdot(Y)).$$

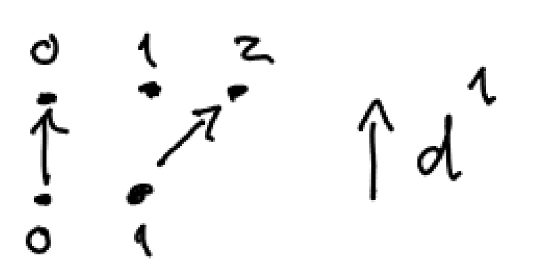
$(\star)$ $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{Sing}_\cdot(Y)) \overset{\text{Yoneda}}{\cong} \mathrm{Sing}_n(Y) \cong \mathrm{Hom}_{\mathrm{Top}}(|X|, Y)$

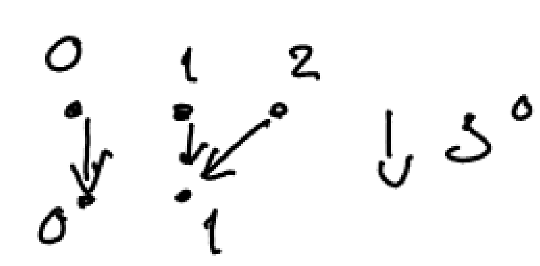
$(✡)$ $\mathrm{colim}(\Delta^\cdot/X \to \Delta \xrightarrow{\Delta^\cdot} \mathrm{sSet}) \cong X$ by the density theorem.

$\square$

Def.: Let $\mathcal{C}$ be a small category. The nerve of $\mathcal{C}$ is
$$N\mathcal{C} = \mathrm{Hom}_{Cat}(-, \mathcal{C}) \in sSet.$$
Note that Δ may be viewed as a subcategory of the category of small categories Cat, where $[n]$ is the free category of the directed linear graph $\overset{0}{\bullet} \to \overset{1}{\bullet} \to \dots \to \overset{n}{\bullet}$.

Claim: The simplex category Δ is generated by the maps $d^i : [n-1] \to [n]$ $(n>0, 0 \leq i \leq n)$ called face maps and $s^i : [n+1] \to [n]$ $(n \geq 0, 0 \leq i \leq n)$ such that





$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+i & \text{if } k \geq i \end{cases} \quad \text{and} \quad s^i(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases}$$

with respect to the following relations:

$\forall i<j \quad d^j d^i = d^i d^{j-1} \;\&\; s^j d^i = d^i s^{j-1}$

$\forall i \leq j \quad s^j s^i = s^i s^{j+1}$

$\forall i > j+1 \quad s^j d^i = d^{i-1} s^j$

$\forall i \quad s^i d^i = s^i d^{i+1} = 1.$

Def.: Let G be a group and $\mathcal{B}G$ be the category with one object and a morphism for each element of G such that composition is given by the group operation of G.

The classifying space of G is $BG = |N(\mathcal{B}G)|$.

More concretely $N(\mathcal{B}G)$ can be described as

$$N_0\mathcal{B}G = \{1\},\ N_1\mathcal{B}G = G,\quad N_2\mathcal{B}G = G\times G, \dots, N_n\mathcal{B}G = G^n, \dots$$

with face maps

$$d_i(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & \text{if } i=0 \\ (g_1,\dots,g_i g_{i+1},\dots,g_n) & \text{if } 0<i<n \\ (g_1,\dots,g_{n-1}) & \text{if } i=n. \end{cases}$$

and degeneracies

$$s_i(g_1,\dots,g_n) = (g_1,\dots,g_i,1,g_{i+1},\dots,g_n).$$

Where we are using d_i for $N(\mathcal{B}(G))(d^i)$ and s_i for $N(\mathcal{B}(G))(s^i)$.

__Def.__: Let $\mathcal{A}$ be an abelian category. There is a functor

$C_*: s\mathcal{A} \longrightarrow Ch_{\geq 0}(\mathcal{A})$ sending a simplicial object $A \in s\mathcal{A}$ to the chain complex $C_*(A)$ with $C_n(A) = A_n$ and boundary maps $\sum_{i=0}^{n} (-1)^i d_i : A_n \longrightarrow A_{n-1}$.

Let $\mathbb{Z}: sSet \to sAb$ denote the post composition with the free abelian group functor.

We recover this way singular homology: let X be a topological space, then $H_*(X;\mathbb{Z}) \cong H_*(C\mathbb{Z}\,Sing_\cdot(X))$.

We also recover group homology: $H_*(G;\mathbb{Z}) \cong H_*(BG;\mathbb{Z})$ which is in fact isomorphic to $H_*(C\mathbb{Z}\,N\mathcal{B}G))$.

This last point is not at all obvious! It relies on the fact that $C\mathbb{Z}N(\mathcal{B}G)$ is quasi isomorphic to $C\mathbb{Z}\,\mathrm{Sing}_\cdot(|N(\mathcal{B}G)|)$.

<u>Def.</u>: Let $X \in sSet$ and $x_0 \in X_0$, the homotopy groups of X are $\pi_n(X, x_0) \cong \pi_n(|X|, |x_0|)$ for all $n \geq 0$.

<u>Def.</u>: The normalized chains functor $N_*: s\mathcal{A} \longrightarrow Ch_{\geq 0}(\mathcal{A})$ for an abelian category $\mathcal{A}$, maps the simplicial object A in $\mathcal{A}$ to the chain complex $N_*(A)$ with $N_n(A) = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \to A_{n-1})$ with the same differential as $C_*(A)$.

$\forall A \in s_R\mathrm{Mod}$ $N_*(A) \subseteq C_*(A)$ and in fact $C_*(A) \cong N_*(A) \oplus D_*(A)$ where $D_*(A)$ is the so called degenerate subcomplex of $C_*(A)$: $D_n(A) = \sum_{i=0}^{n-1} \mathrm{im}(s_i)$.

<u>Claim</u>: $H_k(D_*(A)) \cong 0$ for all $k \geq 0$.

This means, that $N_*(A) \hookrightarrow C_*(A)$ is a quasi isomorphism.

<u>Theorem</u> (Dold-Kan-correspondence): The normalized chains functor $N_*: sAb \longrightarrow Ch_{\geq 0}(Ab)$ is an equivalence of categories with inverse $K_*: Ch_{\geq 0}(Ab) \longrightarrow sAb$

$$C \longmapsto \Big([n] \mapsto \bigoplus_{[n] \twoheadrightarrow [m]} C_m\Big)$$

Furthermore $\pi_*(A,0) \cong H(N_*(A)) \cong H_*(C_*(A))$ $\forall A \in s\mathrm{Ab}$ naturally.

8.6. Cannonical resolutions

<u>Def.</u>: A monad (T,η,μ) on a category $\mathcal{C}$ consists of a functor $T:\mathcal{C}\to\mathcal{C}$, and two natural transformations $\eta: id_{\mathcal{C}} \Rightarrow T$ and $\mu: TT \Rightarrow T$ such that for every $c\in\mathcal{C}$ the diagrams

$$\begin{tikzcd}
TTTc \arrow[r, "T\mu_c"] \arrow[d, "\mu_{Tc}"'] \arrow[dr, phantom, "\circlearrowright"] & TTc \arrow[d, "\mu_c"] \\
TTc \arrow[r, "\mu_c"] & Tc
\end{tikzcd}
\qquad
\begin{tikzcd}
Tc \arrow[r, "T\eta_c"] \arrow[dr, equal] & TTc \arrow[d, "\mu_c"] & Tc \arrow[l, "\eta_{Tc}"'] \arrow[dl, equal] \\
& Tc &
\end{tikzcd}$$

commute. Notation $\mu_{Tc} = \mu Tc$

Monads from adjunctions:

Given an adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ we get natural transformations

$\eta: id \Rightarrow GF$ and $\varepsilon: FG \Rightarrow id$ such that

$\varphi_{c,F(c)}(id_{F(c)}) = \eta_c : c \to GF(c)$ corresponds to $id_{F(c)}$ under $\mathcal{C}(c, GF(c)) \cong \mathcal{D}(F(c), F(c))$

$\varphi^{-1}_{Gd,d}(id_{Gd}) = \varepsilon_d : FG(d) \to d$ corresponds to $id_{G(d)}$ under $\mathcal{D}(FG(d),d) \cong \mathcal{C}(G(d),G(d))$.

The functor GF forms a monad together with η and $G\varepsilon F$:

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "GF\eta"] \arrow[dr, equal] & GFGF \arrow[d, Rightarrow, "G\varepsilon F"] & GF \arrow[l, Rightarrow, "\eta GF"'] \arrow[dl, equal] \\
& GF &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
GFGFGF \arrow[r, Rightarrow, "GF(G\varepsilon F)"] \arrow[d, Rightarrow, "(G\varepsilon F)GF"'] & GFGF \arrow[d, Rightarrow, "G\varepsilon F"] \\
GFGF \arrow[r, Rightarrow, "G\varepsilon F"] & GF
\end{tikzcd}$$

commute for the following reason:

first to see the commutativity of the square, consider that $G\varepsilon : GFG \Rightarrow G$ is a natural transformation and the maps

$GFGFGF \xRightarrow[GFG\varepsilon F]{} GFGF$ is GFG applied to εF and

$GFGF \xRightarrow[G\varepsilon F]{} GF$ is G applied to εF

so the square commutes by naturality of $G\varepsilon$.

To see the commutativity of the triangle we will use the triangle identities:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} GFG \arrow[d, Rightarrow, "G\varepsilon"'] & G \arrow[l, Rightarrow, "\eta G"'] \arrow[dl, equal] \\ G & \end{tikzcd} \quad \text{commute:}$$

Firs consider some $f \in \mathcal{D}(Fc,d)$, then by naturality

$$\begin{tikzcd}
\mathrm{id} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \eta_c \arrow[dd, mapsto] \\
& \mathcal{D}(Fc,Fc) \arrow[r, phantom, "\cong"] \arrow[d, "f_*"] & \mathcal{C}(c,GFc) \arrow[d, "(Gf)_*"] & \\
& \mathcal{D}(Fc,d) \arrow[r, phantom, "\cong"] & \mathcal{C}(c,Gd) & Gf \circ \eta_c \\
f \arrow[rrr, mapsto] & & & \varphi_{c,d}(f)
\end{tikzcd}$$

we see that $Gf \circ \eta_c = \varphi_{c,d}(f)$

and similarly that for any $g \in \mathcal{C}(c,Gd)$

$$\begin{tikzcd}
g \arrow[rrr, mapsto] & & & \varphi^{-1}_{c,d}(g) \\
& \mathcal{C}(c,Gd) \arrow[r, phantom, "\cong"] & \mathcal{D}(Fc,d) & \varepsilon_d \circ Fg \\
& \mathcal{C}(Gd,Gd) \arrow[r, phantom, "\cong"] \arrow[u, "g^*"] & \mathcal{D}(FGd,d) \arrow[u, "(Fg)^*"'] & \\
\mathrm{id} \arrow[rrr, mapsto] \arrow[uuu, mapsto] & & & \varepsilon_d \arrow[uu, mapsto]
\end{tikzcd}$$

so $\varepsilon_d \circ Fg = \varphi^{-1}_{c,d}(g)$

Applying these we get

$$id_{Fc} = \varphi^{-1}_{c,Fc}(\eta_c) = \varepsilon_{Fc} \circ F\eta_c \quad \text{and}$$

$$id_{Gd} = \varphi_{G(d),d}(\varepsilon_d) = G\varepsilon_d \circ \eta_{G(d)}$$

these are exactly the triangle identities. Applying G from the left and F from the right to the triangle identities yields that indeed GF is a monad.

Def.: A comonad in $\mathcal{C}$ consists of a functor $\perp$ and natural transformations $\varepsilon : \perp \Rightarrow id$ and $\delta : \perp \Rightarrow \perp\perp$ such that they form a monad in $\mathcal{C}^{op}$.

If $F \dashv G$, then FG with $\varepsilon : FG \Rightarrow id$ and

$F\eta G : FG \Rightarrow FGFG$ forms a comonad.

Example: Let G be a group and R a ring. There is a morphism of rings $R \to RG$, where RG is the groupring of G. This induces a functor $U : {}_{RG}\mathrm{Mod} \to {}_{R}\mathrm{Mod}$ which is the forgetful functor. This a left adjoint;

$$F(-) = RG \otimes_R -$$

$$\begin{tikzcd} {}_R\mathrm{Mod} \arrow[r, bend left, "F(-) = RG\otimes_R -"] \arrow[r, phantom, "\perp"] & {}_{RG}\mathrm{Mod} \arrow[l, bend left, "U"] \end{tikzcd}.$$

The resulting comonad $FU(-)$ maps the trivial RG-module (on which each $g \in G$ acts by the identity) R, to the RG-module RG.

Simplicial object of a comonad

Let $\mathcal{C}$ be a category $A \in \mathcal{C}$, and $\perp$ a comonad on $\mathcal{C}$. Let $\perp_n A = \perp^{n+1} A$ and define

$$d_i = \perp^i \varepsilon \perp^{n-i} : \perp^{n+1} A \to \perp^n A \quad \forall n \geq 0,\ 0 \leq i < n$$

$$s_i = \perp^i \delta \perp^{n-i} : \perp^{n+1} A \to \perp^{n+2} A \quad \forall n \geq 0,\ 0 \leq i \leq n$$

Claim: $\perp_n A \in s\mathcal{C}$

Def.: An object $A \in \mathcal{C}$ is called $\perp$-projective, if $\varepsilon_A : \perp A \to A$ has a section $f : A \to \perp A$ (i.e. $\varepsilon_A f = id$).

Example: For any ring R we have the free-forgetful adjunction

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & {}_R\mathrm{Mod} \arrow[l, bend left, "U"] \end{tikzcd}$$ which gives a comonad $\perp = FU$.

The $\perp$-projective objects in ${}_R\mathrm{Mod}$ are exactly the projective ones:

If the map $FU(P) \twoheadrightarrow P$ splits, then P is a summand of a free module, thus projective.

If P is projective, then this map shall split according to the universal property of P.

$\perp$-projective objects can also be characterized via a lifting property.

Proposition: Let $\perp$ be a comonad on an abelian category $\mathcal{A}$. If $A \in \mathcal{A}$ is $\perp$-projective, then the augmented simplicial object $\perp_* A \xrightarrow{\varepsilon} A$ is aspherical and the associated augmented chain complex is exact.

$$0 \xleftarrow{\varepsilon} \perp A \xleftarrow{d_0 - d_1} \perp^2 A \leftarrow \perp^3 A \leftarrow \dots$$

Example: Let X be a topological space and $Sh(X)$ the category of sheaves of abelian groups over X.

For $\mathcal{F} \in Sh(X)$ and $x \in X$ the stalk of $\mathcal{F}$ at x is

$\mathcal{F}_x = \operatorname{colim}_{x \in U} \mathcal{F}(U)$.

Furthermore given $A \in Ab$ and $x \in X$, we can form the skyscraper sheaf $x_*(A)$ defined as

$$x_*(A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

$(-)_x \dashv x_*(-)$ and thus $\mathcal{F} \mapsto x_*(\mathcal{F}_x)$ gives a monad.

Let $T(\mathcal{F}) = \prod_{x \in X} x_*(\mathcal{F}_x)$. This also gives a monad the corresponding augmented cosimplicial sheaf $\mathcal{F} \xrightarrow{\eta} T^{*+1}\mathcal{F}$ which in turn provides the so called Godement resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \xrightarrow{\eta} T(\mathcal{F}) \to T^2(\mathcal{F}) \to \dots$$