Mapping cones and mapping cylinders

Let $f: B_\cdot \to C_\cdot$ be a map of chain complexes.

AIM: To ~~forward~~ fit the associated map

$$f_*: H_*(B) \longrightarrow H_*(C)$$ into a long exact sequence.

Def^n: The mapping cone of f is the chain complex $C(f)$ whose degree n part is $B_{n-1} \oplus C_n$ and differentials are given by

$$d(b,c) = (-d(b),\ d(c) - f(b))$$

ie.

$$C(f) = \cdots \to B_n \oplus C_{n+1} \longrightarrow B_{n-1} \oplus C_n \xrightarrow{\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix}} B_{n-2} \oplus C_{n-1} \to \cdot$$

We've a s.e.s

$$\begin{array}{ccccccccc} & & & & (b,c) & \longmapsto & -b & & \\ 0 & \longrightarrow & C & \longrightarrow & \text{Cone}(f) & \longrightarrow & B[-1] & \longrightarrow & 0 \\ & & c & \longmapsto & (0,c) & & & & \end{array}$$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & B_n \oplus C_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n \arrow[r] \arrow[d] & B_{n-1} \oplus C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \oplus C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \, & \, & \, &
\end{tikzcd}$$

Therefore, we've a long exact sequence of ~~Remodules~~ homology groups

$$
\begin{tikzcd}
 & & H_n(B) \arrow[d, phantom, "\cong" rotate=90] \\
 & \arrow[r] & H_{n+1}(B[-1]) \arrow[dll, "\delta"', rounded corners, to path={ -- ([yshift=-2ex]\tikztostart.south) -| (\tikztotarget)}] \\
H_n(C) \arrow[r] & H_n(\mathrm{Cone}(f)) \arrow[r] & H_n(B[-1]) \cong H_{n-1}(B) \arrow[dll, "\delta"'] \\
H_{n-1}(C) & &
\end{tikzcd}
$$

here, the connecting homomorphism

$$\delta = f_*$$

[Recall:

$$
\begin{tikzcd}
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \oplus C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n \arrow[r] \arrow[d] & B_{n-1} \oplus C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & B_n \oplus C_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n+2} \arrow[r] & B_{n+1} \oplus C_{n+2} \arrow[r] & B_{n+1} \arrow[r] & 0
\end{tikzcd}
$$

$c \longmapsto (0, c)$, $(b, c) \longmapsto -b$

$(-b, 0) \dashrightarrow b \in Z_n(B)$

$f(b) \dashrightarrow (d(b), f(b))$

By defn

$$\delta([b]) = [f(b)] = f_*(b)$$

]

ie, we've a l.e.s

$$\cdots \to H_n(B) \xrightarrow{f_n} H_n(C) \longrightarrow H_n(C(f)) \longrightarrow H_{n-1}(B) \xrightarrow{-f_{n-1}} H_{n-1}(C) \to \cdots$$

<u>Corollary</u> A map $f: B \to C$ is a quasi isomorphism iff the mapping cone, $C(f)$ is exact.

<u>Defn</u>: The mapping cylinder ~~cone~~ $cyl(f)$ is a chain complex with the degree n part $B_n \oplus B_{n-1} \oplus C_n$ and whose differentials are given by

$$d(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b'))$$

ie,

$$\to B_{n+1} \oplus B_n \oplus C_{n+1} \to B_n \oplus B_{n-1} \oplus C_n \xrightarrow{\begin{bmatrix} d_B & id_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix}} B_{n-1} \oplus B_{n-2} \oplus C_{n-1} \to$$

Now let's fit $cyl(f)$, $C(f)$ and $f_*: H_*(B) \to H_*(C)$ into a l.e.s of homology groups

We've

$$\begin{tikzcd}
 & & C & & \\
0 \arrow[r] & B \arrow[ur, "f"] \arrow[r] & \mathrm{cyl}(f) \arrow[u, "\beta"] \arrow[r] & \mathrm{cone}(f) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & & C \arrow[u, "\alpha"] \arrow[r] & \mathrm{cone}(f) \arrow[r] & B[-1] \arrow[r] & 0
\end{tikzcd}$$

$(b, b', c) \longmapsto f(b) + c$, $(b, b', c) \longmapsto (b', c)$ (s.e.s)

$b \longmapsto (b, 0, 0)$, $c \longmapsto (0, 0, c)$

$c \longmapsto (0, c)$, $(b, c) \longmapsto -b$ (s.e.s)

With
$\alpha \circ \beta \simeq id_c$
$\beta \circ \alpha \simeq id_{\mathrm{cyl}(f)}$
ie, C and $\mathrm{cyl}(f)$ are chain homotopy

equivalent.
$\therefore$ α & β are quasi isomorphisms.

And we've homology long exact sequence that fit into the following commutative diagram:

$$\begin{tikzcd}
\arrow[r] & H_n(B) \arrow[r] \arrow[d, "\cong"] \arrow[dr, "f_*"] & H_n(\mathrm{cyl}(f)) \arrow[r] \arrow[d, "\cong"] & H_n(\mathrm{cone}(f)) \arrow[r] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, "\cong"] & {} \\
 & H_{n+1}(B[-1]) \arrow[r] & H_n(C) \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & H_n(\mathrm{cone}(f)) \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & H_n(B[-1]) &
\end{tikzcd}$$

1.6 More on Abelian categories

Defn: Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor b/w abelian categories. F is called left (right) exact if, for every short exact sequence

$$0 \to A \to B \to C \to 0,$$

we've $0 \to F(A) \to F(B) \to F(C)$ is exact

$$\left[F(A) \to F(B) \to F(C) \to 0 \text{ is exact} \right]$$

Rmk: F is exact if it is both left and right exact

Prop. Let $\mathcal{A}$ be an abelian category. Then $\mathrm{Hom}_{\mathcal{A}}(M, \_)$ is a left exact functor from $\mathcal{A}$ to Ab

i.e, if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact

then $0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$

$$\phi \; M \to A \longmapsto f \circ \phi$$
$$\psi \longmapsto g \circ \psi$$

is exact in Ab

<u>Proof</u>:

$$g \circ f = 0 \Rightarrow (g \circ f)_+ = 0 \Rightarrow g_+ \circ f_+ = 0$$

$$\Rightarrow \operatorname{Im} f_+ \subseteq \ker g_+$$

let $\alpha \in \operatorname{Hom}(M, A)$ s.

$$f_+(\alpha) = 0 \Rightarrow f \circ \alpha = 0 \Rightarrow f \circ \alpha = f \circ 0$$

$$\Rightarrow \alpha = 0$$

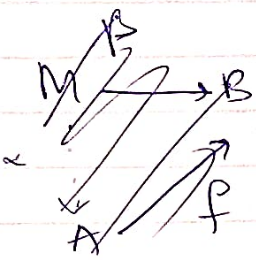
$\ker g_+ \subseteq \operatorname{Im} f_+$

let $\beta \in \operatorname{Hom}(M, B)$ s. $g_+(\beta) = 0$

$$\Rightarrow g \circ \beta = 0$$

$$\Rightarrow \beta(M) \subseteq \ker g = \operatorname{Im} f$$

$$\begin{tikzcd}
A \arrow[r, "\cong"] \arrow[rr, bend left, "f"] & \operatorname{Im} f \arrow[r, hook] & B \\
& & M \arrow[llu, dotted, "\alpha", "\cong"'] \arrow[lu, "\beta"] \arrow[u, "\beta"']
\end{tikzcd}$$



$\cancel{\alpha(m) = f^{-1}(\beta(m))}$

define $\alpha : M \to A$ s. left triangle commutes

then $f \circ \alpha = \beta$

<u>Corollary</u>: $\operatorname{Hom}_{\mathcal{A}}(-, M)$ is a left exact contravariant functor

<u>Proof</u>: $\operatorname{Hom}_{\mathcal{A}}(-, M) = \operatorname{Hom}_{\mathcal{A}^{op}}(M, -)$

Yoneda Embedding :-

Let $\mathcal{A}$ be an additive ~~functor~~ category. Then $\mathcal{A}$ can be embedded in $Ab^{\mathcal{A}^{op}}$ by the functor

$$h : \mathcal{A} \longrightarrow Ab^{\mathcal{A}^{op}}$$
$$A \longmapsto \mathrm{Hom}_{\mathcal{A}}(-, A)$$

Yoneda lemma :-

The Yoneda embedding h reflects exactness

i.e., a sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \text{ in } \mathcal{A} \text{ is exact, provided}$$

that for any M in $\mathcal{A}$, the following sequence is exact

$$\mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow{\alpha^+} \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\beta^+} \mathrm{Hom}_{\mathcal{A}}(M, C)$$
$$\phi \longmapsto \alpha \circ \phi \qquad \psi \longmapsto \beta \circ \psi$$

Proof

Take $M = A$, we get $\beta \circ \alpha = \beta^+ \circ \alpha^+ (id_M)$
$$= 0$$
$$\Rightarrow \mathrm{Im}\,\alpha \subseteq \ker \beta$$

Take $M = \ker\beta$, then $i : \ker\beta \hookrightarrow B$ satisfies $\beta^+(i) = \beta \circ i = 0$

$\Rightarrow \exists\ \sigma \in \mathrm{Hom}(M, A)$ s.t. $i = \alpha^+(\sigma) = \alpha \circ \sigma$

So, $\ker\beta = \mathrm{Im}(i) \subseteq \mathrm{Im}(\alpha)$

2.1 δ-functors

Defn A homological δ-functor b/w abelian categories $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n \geq 0}$ together with morphisms

$$\delta_n : T_n(C) \to T_{n-1}(A)$$

defined for each s.e.s $0 \to A \to B \to C \to 0$ such that

1) $\exists$ a l.e.s

$$\cdots \; T_{n+1}(C) \to T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \to \cdots$$

2) for each morphism of s.e.s

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

H_n: Eg: $CH_{\geq 0}(\mathcal{A}) \to \mathcal{A}$

we've a commutative diagram

$$\begin{tikzcd} T_n(C') \arrow[r, "\delta_n'"] \arrow[d] & T_{n-1}(A') \arrow[d] \\ T_n(C) \arrow[r, "\delta_n"'] & T_{n-1}(A) \end{tikzcd}$$

Defn: A morphism $S \to T$ of δ-functor is a system of natural transformations $S_n \to T_n$ that commute with δ

ie, for each s.e.s

$$0 \to A \to B \to C \to 0$$

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1}(C) \arrow[r, "\delta_n^S"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S_n(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S_n(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S_n(C) \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S_{n-1}(C) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & T_{n+1}(C) \arrow[r, "\delta_n"'] & T_n(A) \arrow[r] & T_n(B) \arrow[r] & T_n(C) \arrow[r, "\delta"'] & T_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

Defn: A homological δ-functor T is universal if for any other δ functor S and a natural transformation $f_0 : S_0 \to T_0$, $\exists$ a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors that extends f_0

$$\begin{tikzcd}
S_0 \arrow[d, "f_0"] & S_1 \arrow[d, dashed, "f_1"] & S_2 \arrow[d, dashed, "f_2"] & S_3 \arrow[d, dashed, "f_3"] \\
T_0 & T_1 & T_2 & T_3
\end{tikzcd}$$

ie, for any s.e.s $0 \to A \to B \to C \to 0$ commutativity

$$\begin{tikzcd}
\arrow[r] & S_2(C) \arrow[r] \arrow[d] & S_1(C) \arrow[r] \arrow[d, "f_1(C)"] \arrow[dr, phantom, "\circlearrowleft"] & S_0(A) \arrow[r] \arrow[d, "f(A)"] & S_0(B) \arrow[d] \\
\arrow[r] & T_2(C) \arrow[r] & T_1(C) \arrow[r] & T_0(A) \arrow[r] & T_0(B)
\end{tikzcd}$$

Eg: We'll see later that $H_* : Ch_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ are universal δ-functors, Derived functors are universal δ functors

2.2 Projective resolutions

An object P in an abelian category $\mathcal{A}$ is projective if it satisfies the following universal lifting property:

Given an epimorphism $g: B \to C$ and a morphism $\gamma: P \to C$, $\exists$ atleast one morphism $\beta: P \to B$ s. $\gamma = g \circ \beta$

$$\begin{tikzcd} & P \arrow[dl, "\exists \beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Eg: Projective modules in Mod R

→ free R-modules are projective modules

→ ~~direct summand of free R-modules are projective module~~

Prop: An R-module is projective iff it is a direct summand of a free R-module.

Proof $\Rightarrow$: let A be a projective R-module. let $F(A)$ be the free module generated by elements of A.

$$\begin{tikzcd} & A \arrow[dl, dashed, "i"'] \arrow[d, "1_A"] & \\ F(A) \arrow[r, "\pi"'] & A \arrow[r] & 0 \end{tikzcd}$$

~~$F(A) = \operatorname{im}(i) \oplus \ker(\pi)$~~

~~We've $\pi \circ i = 1_A$~~

~~$\Rightarrow i$ is injective~~

$$\begin{tikzcd} 0 \arrow[r] & \ker\pi \arrow[r] & F(A) \arrow[r, "\pi"] & A \arrow[r] \arrow[l, bend left, "i"] & 0 \end{tikzcd}$$

ie, $F(A) = A \oplus \ker(\pi)$

$\therefore$ A is direct summand of a free R-module

$\Leftarrow$

claim let $F = A \oplus B$ be free then both A and B are projective.

We'll prove that A is projective

let $\alpha : A \to F$ $\qquad$ $\beta : F \to A$

$x \mapsto (x, 0)$ $\qquad$ $(x, y) \mapsto x$

Note that $\beta\alpha = id_A$

$$\begin{tikzcd} & F \arrow[dl, dashed, "\gamma"'] \arrow[d, shift left, "\beta"] & \\ & A \arrow[u, shift left, "\alpha"] \arrow[d, "f"] & \\ M \arrow[r, two heads, "\pi"'] & N \arrow[r] & 0 \end{tikzcd}$$

Now, let

Since F projective

$\exists\ \gamma$ s.t.

$f \circ \beta = \pi \circ \gamma$

~~$\exists$~~ $\therefore$ let $g = \gamma \circ \alpha$

Then $\pi \circ g = \pi \circ \gamma \circ \alpha = f\beta \circ \alpha = f$. ($\therefore$ A projective)

Defn. $\mathcal{A}$ has enough projectives if for every object A of $\mathcal{A}$ there is an ~~surjection~~ epimorphism $P \to A$ with P projective. Eg: $\mathcal{A} = R\text{-mod}$.

Lemma

M is projective iff $\mathrm{Hom}_{\mathcal{A}}(M,-)$ is an exact functor

ie, iff for every s.e.s

$$0 \to A \to B \xrightarrow{g} C \to 0 \text{ in } \mathcal{A},$$

the sequence of groups

$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \xrightarrow{g_*} \mathrm{Hom}(M,C) \to 0$$

is exact

Proof: ~~enough to show that~~

$\Rightarrow$: Suppose M is projective, it is enough to show that g_* is onto.

Let $\gamma \in \mathrm{Hom}(M,C)$, then

$\exists\ \beta \in \mathrm{Hom}(M,B)$ s.t. $\underset{\substack{\| \\ g_*(\beta)}}{g\circ\beta} = \gamma$

$$\begin{tikzcd} & M \arrow[d, "\gamma"] \arrow[dl, dashed, "\beta"'] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

$\therefore\ g_*$ is onto.

$\Leftarrow$: Let $\gamma : M \to C$ ~~and~~, then $\exists\ \beta \in \mathrm{Hom}(M,B)$

s.t. $g_*(\beta) = \gamma$ $[\because g_* \text{ is onto}]$

ie, $g\circ\beta = \gamma$

$$\begin{tikzcd} & M \arrow[d, "\gamma"] \arrow[dl, dashed, "\beta"'] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

$\therefore\ M$ is projective

<u>Defn</u>: A chain complex P in which each P_n is projective in $\mathcal{A}$ is called a chain complex of projectives.

<u>Defn</u> Let M be an object of $\mathcal{A}$. A <u>left resolution</u> of M is a complex $P_\cdot$ with $P_i = 0$ for $i<0$, together with a map $\varepsilon : P_0 \to M$ so that the augmented complex

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact.

It is a projective resolution if each P_i is projective.

<u>Lemma</u> Every R-module M has a projective resolution. More generally if an abelian category $\mathcal{A}$ has enough projectives, then every object M in $\mathcal{A}$ has a projective resolution.

<u>Proof</u>: Choose a projective P_0 and a surjection $\varepsilon_0 : P_0 \to M$, and set $M_0 = \ker(\varepsilon_0)$. Inductively given a module M_{n-1}, we choose a projective

P_n and a surjection $\varepsilon_n : P_n \to M_{n-1}$.

Set $M_n = \ker(\varepsilon_n)$ and let d_n be the composite

$P_n \to M_{n-1} \to P_{n-1}$. Since $d_n(P_n) = M_{n-1} = \ker(d_{n-1})$,

the chain complex $P.$ is a resolution of M.

"forming a resolution by splicing"

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[dr] & & 0 & & & & \\
M_3 = \ker(\varepsilon_3) \arrow[d] \arrow[ur] & & & M_1 = \ker \varepsilon_1 \arrow[ur] \arrow[dr, "i_1"] & & & & & \\
P_3 \arrow[rr, dashed] \arrow[dr, "\varepsilon_3"'] & & P_2 \arrow[rr, dashed, "d_2"] \arrow[ur, "\varepsilon_2"] & & P_1 \arrow[rr, dashed, "d_1"] \arrow[dr, "\varepsilon_1"] & & P_0 \arrow[r, "\varepsilon_0"] & M \arrow[r] & 0 \\
& M_2 = \ker \varepsilon_2 \arrow[ur] \arrow[dr] & & & & M_0 = \ker \varepsilon_0 \arrow[ur, "i_0"] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$d_2 = i_1 \circ \varepsilon_2$ $\qquad d_1 \circ d_2 = i_0 \circ \underbrace{\varepsilon_1 \circ i_1}_{0} \circ \varepsilon_2$

$d_1 = i_0 \circ \varepsilon_1$ $\qquad = 0$

Comparison theorem

Let $P. \xrightarrow{\varepsilon} M$ be a projective resolution of M and $f' : M \to N$ a map in $\mathcal{A}$. Then for every resolution $Q. \xrightarrow{\eta} N$ of N there is a chain map $f : P. \to Q.$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \varepsilon$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "\exists f_2"'] & P_1 \arrow[r] \arrow[d, "\exists f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\exists f_0"'] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

The chain map f is unique upto chain homotopy equivalence.

Horseshoe lemma

Suppose given a ~~commutative~~ diagram

$$
\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] \arrow[d, dashed] & P_1' \arrow[r] \arrow[d, dashed] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, dashed, "i"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
\cdots \arrow[r, dashed] & P_2'\oplus P_2'' \arrow[r, dashed] \arrow[d, dashed] & P_1'\oplus P_1'' \arrow[r, dashed] \arrow[d, dashed] & P_0'\oplus P_0'' \arrow[r, dashed] \arrow[d, dashed, "\pi"] & A \arrow[r, dashed] \arrow[d, "\pi_A"] & 0 \\
\cdots \arrow[r] & P_2'' \arrow[r] \arrow[d, dashed] & P_1'' \arrow[r] \arrow[d, dashed] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d, dashed] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n$.

Then P_n assemble to form a projective resolution P of A, and the right hand column lifts to an exact sequence of complexes

$$0 \longrightarrow P' \xrightarrow{i} P \xrightarrow{\pi} P'' \longrightarrow 0$$

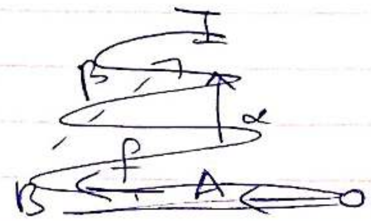
where $i_n : P_n' \longrightarrow P_n$ and $\pi_n : P_n \longrightarrow P_n''$ are the natural inclusion and projection respectively

2.3 Injective resolutions

Defn: An object I in an abelian category $\mathcal{A}$ is injective if it satisfies the following universal lifting property:

Given ~~an injection~~ a monomorphism $f: A \to B$ and a ~~map~~ morphism $\alpha: A \to I$, $\exists$ atleast one ~~map~~ morphism $\beta: B \to I$ such that $\alpha = \beta \circ f$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\exists \beta"] \\ & I & \end{tikzcd}$$



Defn: $\mathcal{A}$ has enough injectives if for every object A in $\mathcal{A}$ there is an ~~injection~~ monomorphism $A \to I$ with I injective.

Lemma: The following are equivalent for an object I in an abelian category $\mathcal{A}$:

i) I is injective in $\mathcal{A}$

ii) I is projective in $\mathcal{A}^{op}$

iii) The contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact, ie, it takes s.e.s in $\mathcal{A}$ to s.e.s in Ab.

<u>Defn</u>: Let M be an object of $\mathcal{A}$. A <u>right resolution</u> of M is a cochain complex $I^{\cdot}$ with $I^i = 0$ for $i<0$ and a map $M \to I^0$ such that the augmented complex

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots$$

is exact.

It is called an injective resolution if each I^i is injective.

<u>Lemma</u>: If the abelian category $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution.

<u>Comparison thm</u>: Let $N \to I^{\cdot}$ be an injective resolution of N and $f': M \to N$ a map in $\mathcal{A}$. Then for every resolution $M \to E^{\cdot}$ there is a cochain map $f: E^{\cdot} \to I^{\cdot}$ liftins f'. The map f is unique upto cochain homotopy equivalence

$$\begin{array}{ccccccccc} 0 \to & M & \to & E^0 & \to & E^1 & \to & E^2 & \to \cdots \\ & f'\downarrow & & \exists f^0 \downarrow & & \exists f^1 \downarrow & & \exists f^2 \downarrow & \\ 0 \to & N & \to & I^0 & \to & I^1 & \to & I^2 & \to \end{array}$$

Defn: A pair of functors $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ are adjoint if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

$$\tau = \tau_{AB}: \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B))$$

here, natural means that for all $f: A \to A'$ in $\mathcal{A}$ and $g: B \to B'$ in $\mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) \arrow[r, "(Lf)^+"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r, "g_+"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) \arrow[r, "f^+"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r, "(Rg)_+"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B'))
\end{tikzcd}$$

We call L the left adjoint and R the right adjoint of this pair

Prop: If

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "L"] & \mathcal{B} \arrow[l, bend left, "R"]
\end{tikzcd}$$

be an adjunction where L is an exact functor and R is an additive functor. Then for any injective object I of $\mathcal{B}$, $R(I)$ is an injective object in $\mathcal{A}$ (R preserves injectives

Dually, if L is additive and R is exact then L preserves projective.

<u>Proof.</u> We must show that $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is exact.

ie, for any s.e.s

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$$\cancel{0 \to \mathrm{Hom}_{\mathcal{A}}(C, R(I)) \to \mathrm{Hom}_{\mathcal{A}}(B, R(I))}$$

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C, R(I)) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(B, R(I)) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A, R(I)) \to 0$$

is exact.

~~Lemma proof~~ It is enough to show that f^* is onto.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(B), I) \arrow[r, "(Lf)^*"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{B}}(L(A), I) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{A}}(B, R(I)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(I))
\end{tikzcd}$$

α

Since L is exact, $L(A) \to L(B)$ is ~~injection~~ monomorphism

Since I is injective

$(Lf)^*$ is onto

$$\begin{tikzcd}
0 \arrow[r] & L(A) \arrow[r] \arrow[d, "\alpha"', "\iota"] & L(B) \\
& I &
\end{tikzcd}$$

and hence f^* is onto.