Introduction to Ext-groups and KK-Theory

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(1) Extensions

Def An extension of A by B is a s.e.s $0 \to B \xrightarrow{i} E \xrightarrow{q} A \to 0$

Ex (i) $0 \to \mathbb{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0$ Toeplitz alg.

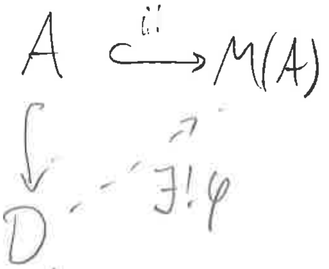
(ii) $0 \to B \to A \oplus B \to A \to 0$

Rmk (Multiplier alg.) $A \subseteq M(A)$ unital C^*-alg. s.t.

A is an essential ideal (i.e., $A \cap I \neq \{0\}$ $\forall I \subseteq M(A)$ ideal) and

$\forall D$ C^*-alg. st. $A \subseteq D$ as an ideal

$\exists! \varphi: D \to M(A)$ with $\ker \varphi = \{d \in D : Ad = \{0\}\}$



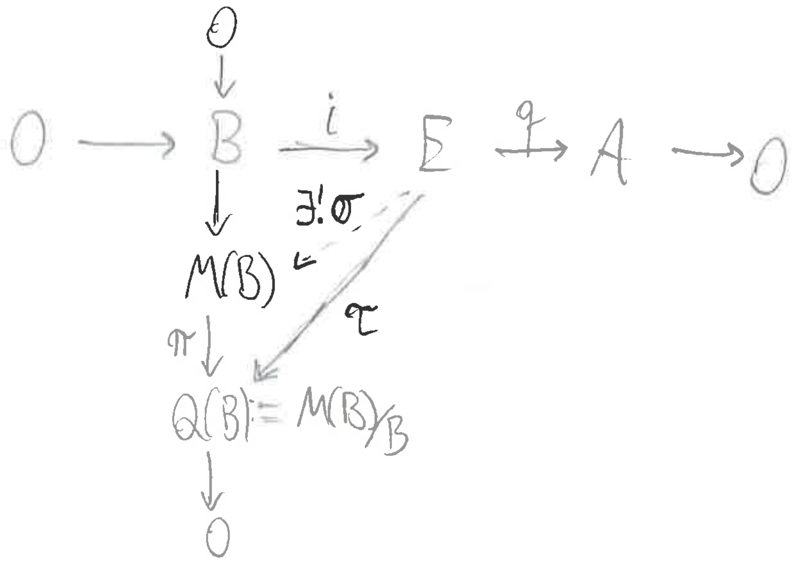
Idea: $M(A)$ is the maximal unital C^*-alg that contains A as an essential ideal.

Ex: $M(C_0(X)) = C(\beta X)$, $M(\mathbb{K}) \cong B(\ell^2\mathbb{N})$, $M(A) = A$ if $1 \in A$.

Def (Busby invariant)

$\tau: A \to Q(B)$

given by $\tau := \pi \circ \sigma$.



Prop $\left\{\begin{array}{c}\text{extensions}\\ \text{of } A \text{ by } B\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{*-hom.}\\ \tau : A \to Q(B)\end{array}\right\}$

Idea: ext. $\longmapsto$ Busby inv.

pullback:

$$\begin{array}{ccc} E & \longrightarrow & A \\ \downarrow & & \downarrow \tau \\ M(B) & \xrightarrow{\pi} & Q(B) \end{array} \quad \longleftarrow\!\!\!| \quad \tau : A \to Q(B)$$

Coro: B unital $\Rightarrow \exists !$ extension
$\hookrightarrow$ ($Q(B)=0$)

Def (Unitary equiv.) $\tau_1, \tau_2 : A \to Q(B)$

$\tau_1 \sim_u \tau_2$ if $\exists\, u \in \mathcal{U}M(B)$ s.t. $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^* \ \forall a \in A$.

▶ Fix B stable ($B \cong B \otimes \mathbb{K}$)

Def $\tau_1 \oplus \tau_2 : A \longrightarrow M_2(Q(B)) \cong Q(B)$

$$a \longmapsto \begin{pmatrix} \tau_1(a) & 0 \\ 0 & \tau_2(a) \end{pmatrix}$$

Rmk $\oplus$ is well-defined up to unitary equivalence.

Prop $\underline{\text{Ext}}(A,B) = \{[\tau] \mid \tau : A \to Q(B) \text{ *-hom}\}$ is a abelian semigroup. ↖ NOT CANCELLATIVE, in general.

Def τ is split (trivial) if $\exists\, \sigma : A \to M(B)$ *-hom st $\tau = \pi \circ \sigma$

Def $\tau_1 \sim \tau_2$ if $\exists\, \tau_1', \tau_2' : A \to Q(B)$ trivial st $\tau_1 \oplus \tau_1' \sim_u \tau_2 \oplus \tau_2'$

Def $\text{Ext}(A,B) := \underline{\text{Ext}}(A,B)/\sim$ $\rightsquigarrow$ ab. monoid.

Rmk τ invertible iff (A separable)

$$\begin{array}{ccc} & \exists \psi \text{ cp} \nearrow & M(B) \\ & \circlearrowleft & \downarrow \pi \\ A & \xrightarrow{\tau} & Q(B) \end{array}$$

Prop. A separable and nuclear $\Longrightarrow$ $\mathrm{Ext}(A,B)$ is a group.
B stable

Rmk: B non-stable $\quad \mathrm{Ext}(A,B) := \mathrm{Ext}(A, B\otimes \mathbb{K})$.

X LCH top. sp.
Rmk $\mathrm{Ext}(X) := \mathrm{Ext}(\underset{\text{nuclear}}{C_0(X)}, \mathbb{C}) \simeq K_1(X)$ (K-homology)

Rmk B sep. $\quad \mathrm{Ext}(\mathbb{C}, B) \cong K_1(B)$.

Def A sep. $\quad K^1(A) := \mathrm{Ext}(A, \mathbb{C})$
$K^0(A) := \mathrm{Ext}(A, C_0(\mathbb{R}))$
K-homology groups (but $K^*(-)$ are contrav.)

Rmk $\mathrm{Ext}(\_\_, \_\_)$ is a bifunctor
(first argument → contravariant, second argument → covariant)

② KK-Theory

All C^*-alg are σ-unital (and separable (?)).

"Def". A Kasparov A-B-module is a triple $(\mathcal{E}, \varphi, F)$
where
(i) $\mathcal{E}$ is a "specific" Hilbert B-module.
(ii) $\varphi : A \longrightarrow \mathcal{B}(\mathcal{E})$ is a "graded" $*$-hom.
(iii) $F \in \mathcal{B}(\mathcal{E})$ s.t. $[F, \varphi(a)]$, $(F^2-1)\varphi(a)$, $(F-F^*)\varphi(a)$ are compact $\forall\, a \in A$.

Idea $\mathbb{E}(A,B) = \{(\mathcal{E}, \varphi, F) : \dots\}$ and $KK(A,B) = \mathbb{E}(A,B)/\sim$

Rmk (Cuntz Picture).

$$KK(A,B) \cong \{(\varphi_0, \varphi_1) \mid \varphi_i : A \longrightarrow M(B \otimes \mathbb{K}) \text{ *-hom; } \varphi_0(a) - \varphi_1(a) \in B \otimes \mathbb{K} \; \forall a \in A\}/\sim$$

↳ quasi-homomorphisms.

Rmk $KK(-,-)$ bifunctor

↳ covar.
↳ contravar.

Prop. $KK^1(A,B) := KK(SA,B) \cong Ext(A,B)$ if A nuclear.

Rmk Kasparov product $KK(A,B) \times KK(B,C) \longrightarrow KK(A,C)$.
(very important)

Prop $KK(\mathbb{C}, B) \cong K_0(B)$, $KK(A, \mathbb{C}) \cong K^0(A)$

Notation $\mathcal{N}$ bootstrap class of separable nuclear C^*-alg.

Def $K_*(A) := K_0(A) \oplus K_1(A)$ and analogously for $K^*(A)$ and $KK^*(A,B)$.

Thm (UCT) $A \in \mathcal{N} \Rightarrow \exists$ a noncannonical split s.e.s.

$$0 \longrightarrow Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A,B) \xrightarrow{\gamma} Hom(K_*(A), K_*(B)) \longrightarrow 0$$

γ ↳ def. using the Kasparov product.

Künneth Thm $A \in \mathcal{N}$, $K_*(A)$ or $K_*(B)$ f.g. $\Longrightarrow \exists$ non-can. split s.e.s.

$$0 \longrightarrow K^*(A) \otimes K_*(B) \xrightarrow{\beta} KK^*(A,B) \longrightarrow Tor^{\mathbb{Z}}_1(K^*(A), K_*(B)) \longrightarrow 0.$$