Chapter 11 - Bott Periodicity

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Recall: $SA = \{f : \mathbb{T} \to A \,/\, f(1) = 0\} \cong C_0((0,1), A)$

$$f \longmapsto \tilde{f}(t) := f(e^{2\pi i t})$$

A unital

Idea: $\forall n \in \mathbb{N} \quad \forall p \in P_n(A)$

Define projection loop $f_p : \mathbb{T} \to \mathcal{U}_n(A)$

$$z \longmapsto zp + (1_n - p)$$

Note that $M_n(\widetilde{SA}) \cong \{ f : \mathbb{T} \to M_n(A) \,/\, f(1) \in M_n(\mathbb{C}1_A) \}$

Idea
$[f_{ij}] \in M_n(\widetilde{SA})$
$\Rightarrow f_{ij} : \mathbb{T} \to A$
Define $f : \mathbb{T} \to M_n(A)$
$z \longmapsto [f_{ij}(z)]$

Hence $f_p \in \mathcal{U}_n(\widetilde{SA})$ (check that $f_p(z) \cdot f_p^*(z) = 1_n \ \forall z$)

Moreover, we can verify that

(i) $f_{p \oplus q} = f_p \oplus f_q \quad \forall \; p, q \in P_\infty(A)$

(ii) $f_0 = 1$.

(iii) $\forall n \; \forall p, q \in P_n(A)$; $p \sim_h q$ in $P_n(A) \Rightarrow f_p \sim_h f_q$ in $\mathcal{U}_n(\widetilde{SA})$

Def The homomorphism $\beta_A : K_0(A) \to K_1(SA)$ given by $\beta_A([p]_0) := [f_p]_1$ is called the Bott map ($p \in P_\infty(A)$)

Prop The Bott map is natural.

Pf: We want

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(\varphi)"] \arrow[d, "\beta_A"'] & K_0(B) \arrow[d, "\beta_B"] \\ K_1(SA) \arrow[r, "K_1(S\varphi)"'] & K_1(SB) \end{tikzcd} \quad (\star)$$

for every $\varphi: A \to B$ unital $*$-hom.

Given $p \in P_\infty(A) \rightsquigarrow$

$$[p]_0 \longmapsto [\varphi(p)]_0 \longmapsto [f_{\varphi(p)}]_1$$

$$[p]_0 \longmapsto [f_p]_1 \longmapsto [\varphi \circ f_p]_1$$

Since $\widetilde{S\varphi}(f_p(z)) = \varphi(zp + 1_n - p) = z\varphi(p) + 1_n - \varphi(p) = f_{\varphi(p)}(z)$, $\forall z \in \mathbb{T}$.

The diagram commutes. $\blacksquare$

What if A is non-unital?

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r] \arrow[d, dashed, "\beta_A"] & K_0(\tilde{A}) \arrow[r, shift left] \arrow[d, "\beta_{\tilde{A}}"] \arrow[dr, phantom, "\circlearrowleft"] & K_0(\mathbb{C}) \arrow[l, shift left] \arrow[r] \arrow[d, "\beta_{\mathbb{C}}"] & 0 \\ 0 \arrow[r] & K_1(SA) \arrow[r] & K_1(S\tilde{A}) \arrow[r, shift left] & K_1(S\mathbb{C}) \arrow[l, shift left] \arrow[r] & 0 \end{tikzcd} \quad (\star\star)$$

split exact

by $(\star)$

$\rightsquigarrow \exists!\ \beta_A : K_0(A) \to K_1(SA)$ making the diagram commute. (Five Lemma).

Notice that, $\beta_A([p]_0 - [s(p)]_0) = [f_p f_{s(p)}^*]_1$, $\forall p \in P_\infty(A)$.

Idea, $\beta_A(-[p]_0)$ must be the unique $[-]_1 \in K_1(SA)$ such that $\underbrace{\beta_A([p]_0 - [p]_0)}_{\beta_A([p]_0) + \beta_A(-[p]_0)} = \beta_A(0) = [1_n]_1 = [f_p f_p^*]_1 = [f_p \oplus f_p^*]_1 = [f_p]_1 + [f_p^*]_1$

Theorem (Bott periodicity) The Bott map $\beta_A : K_0(A) \to K_1(SA)$ is an isomorphism for every C^*-alg A

Rmk: By (★★), it suffices to prove this for A unital,

since $\beta_{\tilde{A}}, \beta_{\mathbb{C}}$ iso $\Rightarrow \beta_A$ iso.

Fix A unital.

Idea: We want to construct an inverse for β_A:

For every $u \in \mathcal{U}_n(\widetilde{SA})$, we need to find a projection $p \in \mathcal{P}_n(A)$ s.t. $[f_p]_1 = [u]_1$.

Def $X \subseteq Y$ locally path connected top. spaces. The inclusion map $\iota : X \hookrightarrow Y$ is said to be a π_0-equivalence if it induces a bijection $\hat{\iota} : \pi_0(X) \to \pi_0(Y)$

connected components!

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d] & Y \arrow[d] \\ \pi_0(X) = X/\sim_h \arrow[r, "\hat{\iota}"] & Y/\sim_h = \pi_0(Y) \end{tikzcd}$$

Rmk: $i : X \hookrightarrow Y$ is a π_0-equiv. iff

(i) $\forall y \in Y \ \exists x \in X$ st. $x \sim_h y$ in Y.

(ii) $x_1, x_2 \in X$, $x_1 \sim_h x_2$ in $Y \Longrightarrow x_1 \sim_h x_2$ in X.

Def $\mathrm{Proj}(n) := \{ f_p \mid p \in \mathcal{P}_n(A) \}$

Rmk If $\mathrm{Proj}(n) \subseteq \mathcal{U}_n(\widetilde{SA})$ was a π_0-equiv. we would be done!

⋮ A lot of techical lemmas

⋮ Really, a lot!

Lemma 11.2.13 $n \in \mathbb{N}$

(i) $\forall\, u \in \mathcal{U}_n(\widetilde{SA})\ \exists\, m \geq n,\ k \in \mathbb{N}\ \exists\, p \in \mathcal{P}_m(A)$

s.t. $(z^k u) \oplus 1_{m-n} \sim_h f_p$ in $\mathcal{U}_m(\widetilde{SA})$.

(ii) $p, q \in \mathcal{P}_n(A)$ and $f_p \sim_h f_q$ in $\mathcal{U}_n(\widetilde{SA})$

$\Longrightarrow \exists\, m \geq n$ and $r \in \mathcal{P}_{m-n}(A)$ s.t. $p \oplus r \sim_h q \oplus r$ in $\mathcal{P}_m(A)$.

Proof (Bott periodicity):

β_A is surjective:

Take $g \in K_1(SA) \Longrightarrow g = [u]_1$ for some $u \in \mathcal{U}_n(\widetilde{SA})$

$\overset{(i)}{\Longrightarrow} \exists\, m \geq n,\ k \in \mathbb{N}\ \exists\, p \in \mathcal{P}_m(A)$ s.t. $(z^k u) \oplus 1_{m-n} \sim_h f_p$ in $\mathcal{U}_m(\widetilde{SA})$.

Then, Whitehead Lemma $\Longrightarrow f_{1_{nk}} = z 1_{nk} \sim_h \underset{z^k 1_n \oplus 1}{z^k 1_n} \oplus 1_{nk-n}$ in $\mathcal{U}_{nk}(\widetilde{SA})$.

Hence $\beta_A([p]_0 - [1_{nk}]_0) = [f_p]_1 - [f_{1_{nk}}]_1 = [z^k u]_1 - [z^k 1_n]_1 =$

$= [u]_1 + [z^k 1_n]_1 - [z^k 1_n]_1 = g.$

<u>Pf</u> β_A is injective:

Take $g \in K_0(A)$ with $\beta_A(g)=0 \rightsquigarrow g=[p]_0-[q]_0$ for $p,q \in P_n(A)$.

$\Longrightarrow [f_p]_1=[f_q]_1 \Longrightarrow f_p \oplus 1_{m-n} \sim_h f_q \oplus 1_{m-n}$ in $\mathcal{U}_m(\widetilde{SA})$
for some $m \geq n$.

Define $p_1 := \operatorname{diag}(p,0) \in P_m(A)$ and $q_1 := \operatorname{diag}(q,0) \in P_m(A)$.

$\Longrightarrow f_{p_1} = f_p \oplus 1_{m-n}$ and $f_{q_1} = f_q \oplus 1_{m-n} \Longrightarrow$

$\Longrightarrow f_{p_1} \sim f_{q_1}$ in $\mathcal{U}_m(\widetilde{SA})$.

Hence

$\xrightarrow{(ii)} p_1 \oplus r \sim_h q_1 \oplus r$ in $P_k(A)$ for some $k \geq m$ and $r \in P_{k-m}(A)$

Thus,

$g = [p]_0 - [q]_0 = [p_1 \oplus r]_0 - [q_1 \oplus r]_0 = 0$. ■

<u>Cor</u> $K_{n+2}(A) \cong K_n(A) \quad \forall n \geq 0$.

<u>Examples:</u>

1. $K_0(C_0(\mathbb{R}^n)) \cong K_n(\mathbb{C}) \cong \begin{cases} K_0(\mathbb{C}) \cong \mathbb{Z}, & n \text{ even}, \\ K_1(\mathbb{C}) \cong \{0\}, & n \text{ odd}. \end{cases}$

   $K_1(C_0(\mathbb{R}^n)) \cong \begin{cases} \{0\}, & n \text{ even} \\ \mathbb{Z}, & n \text{ odd}. \end{cases}$

n-sphere

2. $S^n \cong \mathbb{R}^n \cup \{\infty\}$ $\Longrightarrow K_0(C(S^n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n \text{ even} \\ \mathbb{Z}, & n \text{ odd} \end{cases}$ $K_1(C(S^n)) = \begin{cases} \{0\}, & n \text{ even} \\ \mathbb{Z}, & n \text{ odd}. \end{cases}$

   $C(S^n) \cong \widetilde{C_0(\mathbb{R}^n)}$

Chapter 12 – The Six-Term Exact Sequence

Def $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$ s.e.s. The exponential map $\delta_0 : K_0(B) \to K_1(I)$ is defined by $\delta_0 = \delta_2 \circ \beta_B$

$$K_0(B) \xrightarrow{\beta_B} K_2(B) \xrightarrow{\delta_2} K_1(I).$$

Rmk Let $\bar{\delta}_1 = \theta_I \circ \delta_2$ be the index map w.r.t.

$$0 \to SI \xrightarrow{S\varphi} SA \xrightarrow{S\psi} SB \to 0$$

Then

$$\begin{tikzcd}
K_0(B) \arrow[r, "\delta_0"] \arrow[d, "\beta_B"', "\cong"] & K_1(I) \arrow[d, "\theta_I", "\cong"'] \\
K_1(SB) \arrow[r, "\bar{\delta}_1"'] & K_0(SI)
\end{tikzcd}$$

commutes.

Moral: $\delta_0 = \theta_I^{-1} \circ \bar{\delta}_1 \circ \beta_B$. not really necessary

Theorem (Six-term exact sequence)

$$\begin{tikzcd}
K_0(I) \arrow[r, "K_0(\varphi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \arrow[d, "\delta_0"] \\
K_1(B) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\psi)"] & K_1(I) \arrow[l, "K_1(\varphi)"]
\end{tikzcd}$$

is exact. (*)

long exact seq.

exact

"Proof:

$$\begin{tikzcd}
K_2(A) \arrow[r, "K_2(\psi)"] \arrow[dr, phantom, "\circlearrowleft"] & K_2(B) \arrow[r, "\delta_2"] \arrow[dr, phantom, "\circlearrowleft"] & K_1(I) \arrow[r, "K_1(\varphi)"] \arrow[dr, phantom, "\circlearrowleft"] & K_1(A) \\
K_0(A) \arrow[r, "K_0(\psi)"'] \arrow[u, "\beta_A", "\cong"'] & K_0(B) \arrow[r, "\delta_0"] \arrow[u, "\beta_B", "\cong"'] & K_1(I) \arrow[r, "K_1(\varphi)"'] \arrow[u, equal] & K_1(A) \arrow[u, equal]
\end{tikzcd}$$

$\Downarrow$

$\therefore$ exact

$\Downarrow$

Bott map natural

by def of δ_0

(*) exact at $K_0(B)$ and $K_1(I)$. ∎

Prop. Let $g \in K_0(B)$, $p \in P_n(\tilde{B})$ s.t. $g = [p]_0 - [s(p)]_0$ and $\tilde{\varphi}(a) = p$ for some self-adjoint $a \in M_n(\tilde{A})$.
Then $\exists!\, u \in \mathcal{U}_n(\tilde{I})$ s.t. $\tilde{\varphi}(u) = \exp(2\pi i a)$ and $\delta_0(g) = -[u]_1$.

Rmk
- p exists by the standard picture of K_0
- a exists because is a lift of a self adjoint
- If φ is an inclusion $(I \subseteq A)$, then $\delta_0(g) = -[\exp(2\pi i a)]_1$

Prop δ_0 is natural.

Examples:

(1) K-theory of the Calkin algebra. $(\mathcal{H} \cong \ell^2\mathbb{N})$

$$0 \longrightarrow \mathbb{K} \hookrightarrow B(\mathcal{H}) \longrightarrow Q(\mathcal{H}) \longrightarrow 0$$

$$\begin{tikzcd}
\mathbb{Z} & 0 & 0 \\
K_0(\mathbb{K}) \arrow[u, phantom, "\cong" rotate=90] \arrow[r] & K_0(B(\mathcal{H})) \arrow[u, phantom, "\cong" rotate=90] \arrow[r] & K_0(Q(\mathcal{H})) \arrow[u, phantom, "\cong" rotate=90] \arrow[d, "\delta_0"] \\
K_1(Q(\mathcal{H})) \arrow[u, "\delta_1"] \arrow[d, phantom, "\cong" rotate=90] & K_1(B(\mathcal{H})) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & K_1(\mathbb{K}) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} & 0 & 0
\end{tikzcd}$$

② Toeplitz algebra $\quad S: \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $\delta_n \mapsto \delta_{n+1}$ $\quad \mathcal{T} = C^*(S) \subseteq B(\ell^2\mathbb{N})$

$$0 \longrightarrow \mathbb{K} \xrightarrow{\varphi} \mathcal{T} \xrightarrow{\psi} C(\mathbb{T}) \longrightarrow 0$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, phantom, "\cong" sloped] & & \mathbb{Z} \arrow[d, phantom, "\cong" sloped] \\
K_0(\mathbb{K}) \arrow[r, "K_0(\varphi)"] & K_0(\mathcal{T}) \arrow[r, "K_0(\psi)"] & K_0(C(\mathbb{T})) \arrow[d, "\delta_0"] \\
K_1(C(\mathbb{T})) \arrow[u, "\delta_1"] \arrow[d, phantom, "\cong" sloped] & K_1(\mathcal{T}) \arrow[l, "K_1(\psi)"] & K_1(\mathbb{K}) \arrow[l, "K_1(\varphi)"] \arrow[d, phantom, "\cong" sloped] \\
\mathbb{Z} & & 0
\end{tikzcd}$$

$\delta_0 = 0 \implies K_0(\psi)$ surjective $\implies K_0(\mathcal{T}) \neq 0$.

$K_1(\mathcal{T}) \neq 0 \implies K_1(\psi)$ injective $\implies K_1(\mathcal{T}) \cong m\mathbb{Z}$ for some $m \geq 1$ and $K_1(\psi)$ is just the inclusion

Note $\delta_1 : \mathbb{Z} \to \mathbb{Z}$ is jus $\delta_1(k) = nk$ for some $n \in \mathbb{Z}$. Thus $\delta_1 \circ K_1(\psi) = 0 \implies n = 0$, i.e., $\delta_1 = 0$ $\to$

Oracle : $\delta_1 \neq 0$ $\quad$ ↯ $\quad$ ▷ Rørdam p.168

$K_1(\mathcal{T}) \cong 0 \implies \delta_1$ is injective + codomain $\mathbb{Z} \implies \delta_1$ is iso

$\implies K_0(\varphi) = 0$ and $K_0(\psi)$ iso., i.e., $K_0(\mathcal{T}) \cong \mathbb{Z}$.