Stallings' Graphs and Random Subgroups II

§3.4 Whitehead minimality

Background. Let F be a fin. rank free group. Is there an algorithm st, for evry $H, K \leq F$ fin gen subgroups, determines if there is $\alpha \in \mathrm{Aut}\, F$ st $\alpha H = K$?

Thm. (Gersten). Such an algorithm exists.

Let $S \subseteq F$ be a fin. generating set. The Whitehead automorphisms are a finite family of auto's of F defined as follows. Two types:

- permutations of $S \cup S^{-1}$
- multiplication by some element of $S \cup S^{-1}$ on the left or on the right.

For $H \leq F$, call $\Gamma(H)$ the Stallings graph of H

Lemma (Gersten) If there is $\alpha \in \mathrm{Aut}\, F$ st $|V\Gamma(\alpha H)| < |V\Gamma(H)|$, then there is a Whitehead auto β st $|V\Gamma(\beta H)| < |V\Gamma(H)|$.

Start from H, apply the Whitehead auto's to H to reduce $|V\Gamma(H)|$. This process terminates because:

- Whitehead auto's a fin many
- $|V\Gamma(H)| \in \mathbb{N}$ decreases.

DEF. $H \leq F$ is :

- Whitehead minimal if, for every $\alpha \in \mathrm{Aut}\, F$,
  $$|V\Gamma(H)| \leq |V\Gamma(\alpha H)|$$

- strongly Whitehead minimal if, for every $\alpha \in \mathrm{Aut}\, F$ that not preserve the length, ← word length wrt S
  $$|V\Gamma(H)| < |V\Gamma(\alpha H)|.$$

Let $H, K \leq F$. By argument ↑, we can assume that H and K are Whitehead minimal. If $|V\Gamma(H)| \neq |V\Gamma(K)|$, then H and K must lie in different $\mathrm{Aut}\, F$- orbits.

Thm (Bassino-Nicaud-Weil) For the models :

- uniform distribution of (cyclically reduced) Stallings graphs
- few relators model (restricted to cyclically reduced words)

strict Whitehead minimality is a generic property.

§3.5 Random subgroups of non-free groups.

Let G be a group, $A \subseteq G$ be a fin. generating set. Fix $k \in \mathbb{N}_{\geq 1}$.
Model: for each $n \in \mathbb{N}$, $\mathbb{P}_n$ is the uniform probability on k-tuples of words in $A \cup A^{-1}$ of length $\leq n$.

PROP. (Gilman-Miasnikov-Osin) Let G be a non-elementary hyperbolic group (ie. G is not virtually cyclic). Let $A \subseteq G$ be a fin. gen set, call $\pi: F(A) \longrightarrow G$ the epimorphism. Fix $k \in \mathbb{N}_{\geq 1}$.
Exponentially generically, $\vec{h} \in F(A)^k$ is st $\langle \pi(\vec{h}) \rangle_G$ is

- free
- quasi-convex in G.

non-elementary hyperbolic groups satisfy the "exponentially generic free basis" property.

COR. (GMO) Let G be non-elem hyperbolic, $A \subseteq G$ a fin. gen set. Fix $k \in \mathbb{N}_{\geq 1}$. There is $D \subseteq ((A \cup A^{-1})^*)^k$ exponentially generic and a partial algorithm st, for every $\vec{w} \in D$, for every $z \in (A \cup A^{-1})^*$, it decides if $z \in \langle \vec{w} \rangle_G$.

it does not say wrong answers but an output can be "fail"

outputs: "yes", "no", "fail".

X is exponentially generic if

- it is generic; $\lim_{n \to \infty} \mathbb{P}_n(X) = 1$

- exponentially: $\mathbb{P}_n(X^c)$ is $O(e^{-dn})$ for some $d \in \mathbb{N}_{\geq 1}$.

Idea of the proof. Both PROP and COR rely on two exponentially generic subsets $D_1, D_2 \subseteq ((A \cup A^{-1})^*)^k$

D_1 is the set of k-tuples $\vec{w}$ st $\langle \vec{w} \rangle_G \cong F_k$

D_2 is the set of k-tuples $\vec{w} = (w_1, \dots, w_k)$ st

$$|w_i^{\pm\varepsilon} \cdot w_j^{\mp\varepsilon}| > \max\{|w_i|, |w_j|\} + 2\delta$$

hyperbolicity of G

(for $i \neq j$) and $\varepsilon \in \{\pm 1\}$.

In the paper, they construct an exponentially generic $C \subseteq ((A \cup A^{-1})^*)^k$ and show that

$$C \subseteq D_1 \quad \text{and} \quad C \subseteq D_2.$$

$$X = ((A \cup A^{-1})^*)^k$$

$\mathbb{P}_n : P(X) \to [0,1]$ uniform probability on $(A \cup A^{-1})^k_{\leq n}$

$$D_1 = \{ \vec{w} \in X : \langle \vec{w} \rangle_G \cong F_k \} \subseteq X$$