RANDOM SUBGROUPS OF FREE GROUPS

Notation A finite alphabet on m letters, $F(A)$ Free group on A

Def A-Graph = finite directed graph, with edges labelled by letters of A

An A-Graph is → reduced if no two outgoing (resp. ingoing) arrows have the same label

AKA NO $\bullet \overset{a}{\nearrow} \overset{a}{\rightarrow}$, NO $\overset{a}{\searrow} \bullet \overset{a}{\nwarrow}$

→ cyclically reduced if moreover there are no valence-one vertices

→ rooted at a vertex p if every vertex except p has valence ≥ 2

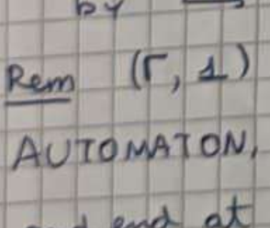
Rem Given Γ ~~[illegible]~~, one can always cyclically reduce Γ to a graph $K(\Gamma)$, as follows

$$\Gamma \xrightarrow[\text{STALLINGS FOLDING}]{} \Gamma' \xrightarrow[\text{remove leaves}]{} K(\Gamma)$$

STALLINGS FOLDING:

$\forall$ $\overset{a}{\nearrow}\overset{a}{\rightarrow}$ is replaced by $\overset{a}{\rightarrow}$

Rem $(\Gamma, 1)$ rooted reduced graph can be seen as an AUTOMATON, accepting the language of words that start and end at 1. This language is a SUBGROUP, which can be computed by
- taking a spanning tree T_Γ
- for every edge e outside T_Γ, consider the loop based at 1 obtained by getting to one endpoint of e inside T_Γ, going through e, and going back inside T_Γ



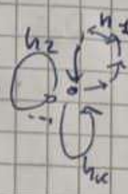
Es $T_\Gamma = \xrightarrow{a}\xrightarrow{a}\overset{b}{\nearrow}$

$H_\Gamma = \langle b, a^2 b^2 a^{-1} \rangle$

__Conversely__ given a finitely generated subgroup $H \leq F(A)$
$\exists!$ rooted reduced graph $\Gamma(H)$ accepting H, called
the STALLINGS GRAPH of H.

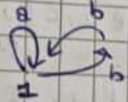
__Construction__ $H = \langle h_1, \dots, h_k \rangle$

$\Gamma(H)$ = obtained by folding the rose with labels $h_1, \dots, h_k$



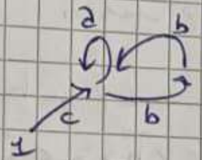
__Rem__ H, K are conjugate $\iff$ $K(\Gamma_H) = K(\Gamma_K)$
$\iff$ $\Gamma(H)$ and $\Gamma(K)$ differ by the initial segment containing 1

__Es__ $\langle a, b^2 \rangle =$



~~$\langle\langle a, ab^2 a^{-1} \rangle\rangle$~~

$\langle cac^{-1}, cb^2c^{-1} \rangle =$



__Facts about H you can read from $\Gamma(H)$__

E = edges of $\Gamma(H)$
V = vertices of $\Gamma(H)$

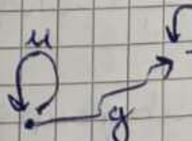
① $\mathrm{rank}(H) = E - V + 1$ since $\pi_1(\Gamma(H)) = H$

② H has finite-index iff $\Gamma(H)$ is a COVERING of the rose, i.e. the degree of every vertex is $2m$, i.e. $E = mV$. If so, the index is V

→ __Proof__ finite coverings of the rose $\iff$ finite subgroups.

③ H malnormal $\iff$ $\nexists\, u \in A^*$ labelling two loops based at distinct vertices ⊛

→ __Proof__ $\Rightarrow$ ~~If~~



$\circlearrowleft u \iff$ ~~then~~ $u, g^{-1}ug \in H$
$\iff H \cap gHg^{-1} \supseteq \langle u \rangle$

④ H pure $\iff$ $\forall u \in A^*$, if u^n labels a loop, then u labels a loop.

→ __Proof__ straightforward.

Now $\vec{h}$ reduced tuple, $\vec{h}^{\pm} = h \cup h^{-1}$, ...

$lcp(\vec{h})$ = lenght of longest common prefix of two words in $\vec{h}^{\pm}$

$H = \langle \vec{h} \rangle$

Def $\vec{h}$ has the CENTRAL TREE PROPERTY if $lcp < \frac{\min h}{2}$

↳ "some sort of initial small cancellation condition"

The Stallings graph for H looks like

central tree of shared prefixes

one arc for every h_i which is NON-TRIVIAL since $2\,lcp < |h_i|$

Consequences
- $\vec{h}$ is a free basis for H (there are no redundant gen.)
- If $\langle \vec{h} \rangle = \langle \vec{g} \rangle$ and both have the CTP, then the Stallings graph is the same (it only depends on H) and therefore $\vec{h}^{\pm} = \vec{g}^{\pm}$ as each word corresponds to one of the arcs.

AKA Up to reordering and taking inverses, there is a unique "best" basis for H

Lem If $lcp(\vec{h}) < \frac{\min(h)}{3}$ and every word of lenght at least $\frac{1}{2}(\min(h) - 2\,lcp)$ appears at most once in $\vec{h}^{\pm}$, then H is MALNORMAL

Proof Check that $\Gamma(H)$ satisfies the aforementioned property (A) on words not labelling two loops in $\Gamma(H)$.

First two models of random subgroups

$R_{\leq \ell}$ = reduced words of lenght $\leq \ell$

- density model: $0 < d < 1$, select $\vec{h}$ of cardinality $|R_{\leq \ell}|^d$
- few generators model: $k \geq 1$, select $\vec{h}$ of cardinality k

__Thm (3.2)__ In the density model, the following hold wop:

- $\vec{h}$ has CTP $\iff d < 1/2$
- If $d < 1/4 \Rightarrow (*) \Rightarrow H$ is malnormal and freely generated by $\vec{h}$

Having CTP (and the condition $*$) pass to sub-tuples, so we get the same in the few generators model:

__Thm 3.3__ In the K-generators model (for $\forall K$), the following hold wop:

- $\vec{h}$ has CTP
- H is malnormal and freely generated by $\vec{h}$

__Proof in the few generator model__ follows from the next lemma, from which one can also derive $C'(\lambda)$ - small cancellation:

__Lem (Arzhantseva, Ol'shanskii)__ __Fix__ $0 < \alpha < 1/2$, $2\alpha < \beta < 1$, $0 < \lambda < \frac{1}{2}$, $k \geq 0$.

Then a random k-tuple $\vec{h}$ from $R_{\leq \ell}$ satisfies wop:

ⓐ $\min \vec{h} > \beta \ell$ → most words are of full lenght

ⓑ $\mathrm{lcp}\, \vec{h} < \alpha \ell$ → most prefixes are very short

ⓒ no word of lenght $\geq \lambda \ell$ appears twice in $\vec{h}^{\pm}$ → practically $C'(\lambda)$

__Proof__ ⓐ $\#R_{\leq \ell} = 1 + \sum_{k=0}^{\ell-1} 2m(2m-1)^{k} = 1 + 2m\left(\frac{(2m-1)^{\ell}-1}{2m-2}\right)$

$\#R_{\leq \beta\ell} = 1 + 2m\left(\frac{(2m-1)^{\beta\ell}-1}{2m-2}\right)$

$$\left(\frac{R_{\leq \beta\ell}}{R_{\leq \ell}}\right)^{k} \xrightarrow[\ell \to +\infty]{} 0$$

ⓑ By ⓐ, assume all words of lenght $\sim \ell$

- if $|u| > \ell\lambda$ appears in some h_i as a prefix and u^{-1} as a suffix, then there are $|u|$ free letters less in $\vec{h}$ → there are $\leq 2m(2m-1)^{\ell-1}(K-1) + 2m(2m-1)^{\ell-1-\lambda\ell}$ tuples with this repetition
- if $|u| > \ell\lambda$ appears as a prefix in h_i and h_j then again there are $\leq 2m(2m-1)^{\ell-1}(k-1) + 2m(2m-1)^{\ell-1-\lambda\ell}$ tuples like this

$\Rightarrow$ the sum of all these tuples $/ \#R_{\leq \ell}^{k} \longrightarrow 0$

ⓒ is proven like ⓑ, with a little extra care.

~~Rem Since tuples with the CTP correspond to subgroups BISE~~

Rem Since most subgroups of $F(A)$ admit a basis with CTP, and since any two such bases are essentially the same $(\vec{h}^{\pm} = \vec{g}^{\pm})$, one can consider the probability of picking a RANDOM SUBGROUP on some generators, rather than a random tuple; this is the same as what we did for one-relator quotients

ANOTHER MODEL OF RANDOMNESS

→ similar to Erdos-Renij but we want them oriented and not simplicial

Every subgroup corresponds to a unique Stallings graph, so one can do the following:

- fix V;
- extract a rooted reduced tree on V vertices, uniformly at random;
- look at the behaviour as $V \to +\infty$.

How to count graphs?

Look at the edges labelled by $a \in A$. Since Γ must be reduced, these edges correspond to a PARTIAL BIJECTION $\{1, \dots, V\} \circlearrowleft$ (aka a bijection between two subsets of $\{1, \dots, V\}$)

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ 2 & & 1 \\ \vdots & \longrightarrow & 3 \\ V & \longrightarrow & V \end{array}$$

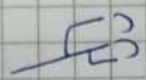
$$PB_V^k = \binom{V}{k}^2 k!$$

partial bijections between two subsets of k elements — choose the subsets — shuffle them

$$PB_V = \sum_{k=0}^{V} PB_V^k$$

So one picks m Partial bijections, one for each $a \in A$, choosing k_a From the distribution $\left(\frac{PB_V^0}{PB_V}, \dots, \frac{PB_V^V}{PB_V} \right)$, and then checks if the graph is rooted and reduced.

K-generators model

- $\mathbb{E}(\text{rank}) = K$, since $\vec{h}$ will have the CTP
- graph looks like 

  → it is very "sparse"
- H has infinite index & (see below)
- H is malnormal
- $F(A)/\langle\langle H \rangle\rangle$ is infinite (non elem hyp)

graph model

- $\mathbb{E}(\text{rank}) = (m-1)V - m\sqrt{V} + 1 \xrightarrow[V \to +\infty]{} \infty$
- graph has way more loops

  → it is very "dense" with edges
- $\mathbb{E}[E] = \mathbb{E}[rk] + V - 1 = m(V - \sqrt{V}) < m|V|$

  $\Rightarrow$ $\mathbb{E}[E]$ is not maximum

  $\Rightarrow$ H will not be finite-index.

  (idea: it is very rare that ∀ partial bijection is actually a bijection)
- very likely that a partial bijection has a loop $1 \to 2 \to 3$

  but no loop of lenght 1

  $\Rightarrow$ H is not pure, hence not malnormal
- very likely that a partial bijection has TWO LOOPS OF COPRIME LENGHT

  $\Rightarrow$ $\langle\langle H \rangle\rangle$ contains every generator

  $\Rightarrow$ $\boxed{F(A)/\langle\langle H \rangle\rangle = 1}$

$\Rightarrow$ the random graph is a terrible model for random groups.