Gauge field theory

Gauge theory and Higgs field

$D+1$ dim Mink space

one-form $A = A_\mu dx^\mu$

$A_\mu = A_\mu^a T^a$

T^a: generator of Lie group G

covariant derivative:

$D = d + A$ connection

Lie group G

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA + A \wedge A$$

field strength
"curvature"

$$F_{\mu\nu} = F_{\mu\nu}^a T^a$$

$$F_{\mu\nu} = [D_\mu, D_\nu]$$

action;

$$S_{YM} = -\frac{1}{2g_s^2} \int \mathrm{Tr}(F \wedge * F)$$

θ-term

$$S_\theta = \frac{\theta}{8\pi^2} \int \mathrm{Tr}(F \wedge F) \sim \int \mathrm{Tr}(d(F \wedge A))$$

topological term

Invariant under gauge transf $\quad g = g(x^\mu) \in G$

$$A' = g A g^{-1} - dg \cdot g^{-1}$$

$$F' = g F g^{-1}$$

Bianchi $\quad DF = dF + [A, F] = 0$

EoM: $\quad D * F = 0$

Higgs field: $\quad \phi : \mathbb{R}^{D+1} \longrightarrow \mathfrak{g}$ $\quad$ in the adjoint rep

$$D\phi = d\phi + [A, \phi]$$

$$\phi' = g \phi g^{-1}$$

Hodge star: $\quad * : \Lambda^p \longrightarrow \Lambda^{D+1-p}$

$$*\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\right) = \frac{1}{(D+1-p)!} \varepsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_{D+1}} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_{D+1}}$$

$$[T_a, T_b] = \underbrace{f_{abc}}_{\text{structure constants}} T_c \qquad a, b, c = 1, \ldots, \dim G$$

usually $G = SU(N)$

If $\quad D=4 \qquad (*F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$

- $*F = F$ self-dual (SD)
- $*F = -F$ anti-self-dual (ASD)

Scaling argument

fields: (A, ϕ)

energy: $$E = \int_{\mathbb{R}^D} d^D x \left(|F|^2 + |D\phi|^2 + U(\phi) \right) =$$

$$= E_F + E_{D\phi} + E_U$$

$$\phi_{(c)}(x) = \phi(cx) \qquad A_{(c)}(x) = c\,A(cx)$$

$$D_{(c)}\phi_{(c)} = c\,D\phi(cx) \qquad F_{(c)}(x) = c^2 F(cx)$$

$$E_{(c)} = \frac{1}{c^{D-4}} E_F + \frac{1}{c^{D-2}} E_{D\phi} + \frac{1}{c^D} E_U$$

Minimize $E_{(c)}$: $\left.\frac{dE_{(c)}}{dc}\right|_{c=1} = 0 \Rightarrow (D-4)E_F + (D-2)E_{D\phi} + DE_U = 0$

$E_{[C]}$ can be stationary provided that $0 \le D \le 4$

- $D=1$ Gauged kinks
- $D=2$ Vortices with $E_F = E_V$
- $D=3$ Non-abelian monopoles with $E_F = E_{D\phi}$
- $D=4$ instantons in pure YMs

Dirac monopole and flux quantization

Maxwell theory: $U(1)$ gauge theory on $\mathbb{R}^{3,1}$

$$F = E_i \, dx^i \wedge dt + \frac{1}{2} \varepsilon_{ijk} B_i \, dx^j \wedge dx^k$$

$$J = J_\mu \, dx^\mu$$

EoM: $\quad d * F = * J$

Bianchi $\quad dF = 0 \quad \leftarrow$ no magnetic monopoles

When $J = 0$: $\quad F \to *F$ is a symmetry between the electric and magnetic fields

"electric - magnetic duality"

$F = dA \Rightarrow dF = 0 \qquad$ How to avoid this?

$\mathbb{R}^3 - \{0\}$ with coord patches U_+ and U_-

U_+: $z > -\varepsilon \qquad U_-$: $z < \varepsilon$

$$A_\pm = \frac{g}{4\pi r} \cdot \frac{1}{z \pm r} \left(x\, dy - y\, dx \right) =$$

$$= \frac{g}{4\pi} \left(\pm 1 - \cos\theta \right) d\phi \qquad g = \text{const}$$

$$A_+ = A_- + \frac{g}{2\pi}\, d\phi$$

$$F = dA_\pm = \frac{g}{4\pi} \sin\theta \, d\theta \wedge d\phi$$

$$\Rightarrow \vec{E} = 0 \qquad \vec{B} = \frac{g\vec{r}}{4\pi r^3}$$

F is closed but not exact $\Rightarrow$ magnetic monopole

Magnetic flux through a sphere:

magnetic charge

$$Q = \oint_{S^2} F = \int_{U_+} dA_+ + \int_{U_-} dA_- = \oint_C (A_+ - A_-) = \frac{g}{2\pi}$$

Dirac quantisation (Tong)

Not any g is compatible with QM!

Electric charge e, adiabatically transported along closed path C in background $\vec{A}(\vec{x}, t)$

$$\psi \to e^{ie\frac{\alpha}{\hbar}} \psi \qquad \alpha = \oint_C \vec{A} \cdot d\vec{x} = \int_D d\vec{S} \cdot \vec{B}$$

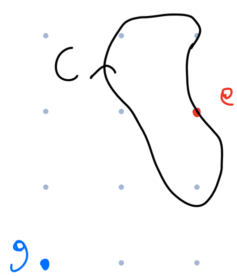
example of a Berry phase

phase difference → Aharonov–Bohm effect

93 (Aharonov) ; Bohm: αν ζούσε 109

Nobel πότε?

In the presence of a magnetic monopole g:



$$\alpha = \int_S d\vec{S} \cdot \vec{B}$$

$$\vec{B} = \frac{g\,\vec{r}}{4\pi r^3}$$

S^2: sphere of radius R

S: region of S^2 with solid angle Ω

$$\alpha = \frac{\Omega g}{4\pi}$$

S' complement of S

$$\Omega' = 4\pi - \Omega$$

$$\alpha' = -\frac{(4\pi - \Omega)}{4\pi}\, g$$

We want: $e^{ie\alpha/\hbar} = e^{ie\alpha'/\hbar}$ (Anomaly cancelation in the SM)

$$eg = 2\pi\hbar\, n \qquad n \in \mathbb{Z}$$

Dirac quantization condition

smallest magn charge → quantum of flux $\Phi_0 = \dfrac{2\pi\hbar}{e}$ (tell them about the Witten effect)

Non-abelian monopoles ('t Hooft - Polyakov monopoles)

$G = SU(2)$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - U(\phi)$$

$$U(\phi) = \frac{1}{4} c \left(|\phi|^2 - v^2 \right)^2 \qquad |\phi|^2 = \phi^a \phi^a$$

Finite energy configurations:

$$\Rightarrow \quad r \to \infty: \quad |\phi| \to v \qquad D_\mu \phi \to 0$$

$$F_{\mu\nu} \to 0$$

EoM:
$$\left(D_\mu F^{\mu\nu} \right)^a = -\varepsilon^{abc} \phi^b \left(D^\nu \phi \right)^c$$

$$\left(D_\mu D^\mu \phi \right)^a = -c \left(|\phi|^2 - v^2 \right) \phi^a$$

We are looking for solitons:

Non-singular, static, finite energy solutions of the classical field equations

~ Solitons

Topology of monopoles

Pick gauge $A_0 = 0$, set $v = 1$

$$\hat{\Phi} = \frac{\Phi}{|\Phi|} \qquad \hat{\Phi}_\infty = \lim_{r \to \infty} \Phi$$

map from S^2_∞ to the unit two sphere in the Lie algebra

topologically classified by its degree N

relate to outward magnetic flux at infinity

Non-abelian monopoles:

fields are smooth everywhere in $\mathbb{R}^3$

unlike the Dirac monopole

But: $SU(2)$ is broken down to $U(1)$ at infinity by $\hat{\Phi}$

$\Rightarrow$ looks like a Dirac N monopole when viewed from a distance

The non-vanishing part of F:

$$B_k^{(m)} = \frac{1}{g} \varepsilon_{ijk} \hat{F}_{ij}^a \hat{\phi}^a$$

$$A_i^a = -\varepsilon^{abc} \partial_i \hat{\phi}^b \hat{\phi}^c + k_i \hat{\phi}^a$$

$$F_{ij}^a = 2 \varepsilon^{abc} \partial_i \hat{\phi}^b \partial_j \hat{\phi}^c - \left(\varepsilon^{pqr} \partial_i \hat{\phi}^p \partial_j \hat{\phi}^q \hat{\phi}^r \right)$$

Magnetic charge: $Q = \int_{S^2_\infty} B_k^{(m)} n^i d^2S = 4\pi N$

$N = \deg(\hat{\phi}_\infty)$ N: monopole number

unit of the monopole charge $= 4\pi$

Bogomolny-Prasad-Sommerfeld (BPS) limit

Consider the limit: $c = 0$, $|\phi_\infty| = 1$

$$B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$$

Theorem: The energy of a non-abelian magnetic monopole is bounded from below:

$$E \geq 4\pi |N|$$

saturated if $*_3 F = D\phi$

$\downarrow$

BPS monopoles

$*_3 : \Lambda^k \to \Lambda^{3-k}$

$$*_3 \, dx^k = \frac{1}{2} \varepsilon^{ijk} \, dx^j \wedge dx^k$$

Yang-Mills equations and instantons

Def: Instantons are non-singular solutions of classical equations of motion in Euclidean space whose action is finite

$$S = -\int_{\mathbb{R}^4} \mathrm{Tr}\left(F \wedge * F\right)$$

EoM: $D * F = 0$

$r \to \infty$: $F_{\mu\nu}(x) \sim \frac{1}{r^3}$ $\quad A_\mu(x) \sim -\partial_\mu g \, g^{-1} + \mathcal{O}(r^{-2})$

$$g : S^3_\infty \to SU(2)$$

Chern and Chern-Simons forms

F take values in $su(n)$

Chern class:

$$C(F) = \det\left(1 + \frac{i}{2\pi} F\right) = 1 + C_1(F) + C_2(F) + \dots$$

$C_p(F)$ pth Chern form

2p-form

for $G = SU(2)$

$$C_1(F) = \frac{i}{2\pi} \operatorname{Tr}(F) = 0$$

$$C_2(F) = \frac{1}{8\pi^2}\Big[\operatorname{Tr}(F\wedge F) - \operatorname{Tr}(F)\wedge \operatorname{Tr}(F)\Big] =$$

$$= \frac{1}{8\pi^2} \operatorname{Tr}(F\wedge F)$$ $\sim$ theta term

In any dim: $\quad dC_2 = \frac{1}{4\pi^2} \operatorname{Tr}(dF \wedge F) = 0$

On $\mathbb{R}^D$ $\quad dC_2 = 0 \Rightarrow \quad C_2 = dY_3$

Y_3: Chern-Simons three-form

$$Y_3 = \frac{1}{8\pi^2} \operatorname{Tr}\left(dA\wedge A + \frac{2}{3}A^3\right)$$

In $D=4$:

Chern number $c_2 = \int_{\mathbb{R}^4} C_2 = \frac{1}{8\pi} \int d \operatorname{Tr}\left(F\wedge A - \frac{1}{3}A^3\right)$

$$c_2 = -\frac{1}{24\pi^2}\int_{S^3_\infty} \mathrm{Tr}(A^3) =$$

$$= \frac{1}{24\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left[\left((dg)g^{-1}\right)^3\right] = \deg(g) \in \mathbb{Z}$$

Minimal action solutions and ASD condition

Theorem: The YM action within a given topological sector:

$$c_2 = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F \wedge F) > 0$$

is bound from below:

$$S_{YM} \geq 8\pi^2 c_2$$

saturated when: $F = -*F$

ASD

Bogomolny bound

for $c_2 \le 0 \Rightarrow$ SD $F = *F$

The ASDYM field satisfying the bc:

$r \to \infty \quad F_{\mu\nu}(x) \sim O(r^{-3}) \qquad A_\mu(x) \sim -\partial_\mu g\, g^{-1} + O(r^{-2})$

in 4d are called instantons!

$k = -c_2 \Rightarrow$ Instanton number

Ansatz for ASD fields

Introduce: $\sigma_{ab} = \varepsilon_{abc} T_c \qquad \sigma_{a4} = -\sigma_{4a} = T_a$

T_a basis of $su(2)$

$$\frac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}\,\sigma_{\kappa\lambda} = \sigma_{\mu\nu} \Rightarrow \text{SD}$$

Proposition:

$$A = \sigma_{\mu\nu}\frac{\partial_\nu \rho}{\rho}\,dx^\mu$$ satisfies the ASDYM equations iff:

$$\Box \rho = 0$$

$\rho = r^{-2}$ is pure gauge $\Rightarrow$ $F = 0$

Jackin - Nohl - Rebbi N-instanton solution:

$$\rho = \sum_{p=0}^{N} \frac{\lambda_p}{|x - x_p|^2}$$