Lecture 2

Field theory

$\mathbb{R}^{1,D}$ $\qquad x^\mu = (t, \vec{x})$ $\qquad (+ - - \dots -)$

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - d\vec{x}^2$$

N scalars: $\phi : \mathbb{R}^{D+1} \longrightarrow Y \subset \mathbb{R}^N$ $\qquad \phi^a, \; a = 1, \dots, N$

Lagrangian density: $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a)$

$$S = \int_{\mathbb{R}^{D,1}} \mathcal{L}\, d^{D+1}x \qquad \frac{\partial \mathcal{L}}{\partial \phi^a} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} = 0$$

for $\quad \mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - U(\phi)$

$$\Rightarrow \qquad \partial^\mu \partial_\mu \phi^a = - \frac{\partial U}{\partial \phi^a}$$

Symmetries: Transformations that leave the action (path integral)) invariant

Continuous global symmetries

Noether's theorem

Field transformation; $\delta \phi^a = \varepsilon W^a_{(x)}$

$\delta \mathcal{L} = \varepsilon \partial_\mu B^\mu$

$\varepsilon \longrightarrow \varepsilon(x)$

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu (\varepsilon W^a) + \frac{\delta \mathcal{L}}{\partial \phi^a} \varepsilon W^a =$$

$$= \partial_\mu \varepsilon \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} W^a_{(x)} + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu W^a + \frac{\partial \mathcal{L}}{\partial \phi^a} W^a_{(x)} \right]}_{= \partial_\mu B^\mu} \varepsilon$$

$$\delta \mathcal{L} = \partial_\mu \varepsilon \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} W^a(x) = \partial_\mu \varepsilon \, \tilde{J}^\mu$$

$$\tilde{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} W^a$$

$$\delta S = \int d^d x \, \delta \mathcal{L} = \int d^d x \left[\partial_\mu \varepsilon \, \tilde{J}^\mu + \partial_\mu B^\mu \varepsilon \right] = - \int d^d x \, \varepsilon \, \partial_\mu (\tilde{J}^\mu - B^\mu) =$$

$$= - \int d^d x \, \varepsilon \, \partial_\mu J^\mu$$

$$\Rightarrow \quad \partial_\mu J^\mu = 0 \quad \left(\begin{array}{c} \text{on-shell} \\ \delta S = 0 \end{array} \right)$$

Typically in QFT we examine correlation functions in the vacuum state $\rightsquigarrow$ perturbation theory

However here we will study a different aspect of QFT.

We examine a particular family of field configurations: solitons $\rightsquigarrow$ non-perturbative

"soliton" has a different meaning in field theory

Previously

Soliton: localized lump of energy that survives scattering with other solitons

Definition: Solitons are non-singular, static,
(in field theory) finite energy solutions of the classical field equation

At the quantum level solitons correspond to localized extended objects (quasiparticles)

Consider a scalar field on 2-dim spacetime

$$L = \int_{\mathbb{R}} \left(\frac{1}{2} \phi_t^2 - \frac{1}{2} \phi_x^2 - U(\phi) \right) dx = T - V$$

$$T = \int_{\mathbb{R}} \frac{1}{2} \phi_t^2 \, dx \qquad V = \int \left[\frac{1}{2} \phi_x^2 + U(\phi) \right] dx$$

EoM: $$\phi_{tt} - \phi_{xx} = -\frac{dU}{d\phi}$$

$$U(\phi) \geq 0 \qquad U(\phi_1) = U(\phi_2) = 0$$

Kink solution

$\phi \cong \phi_1$ as $x \to -\infty$ $\qquad$ $\phi \cong \phi_2$ as $x \to +\infty$

static field $\Rightarrow$ $\phi_t = 0$

$$\phi_{xx} = \frac{dU}{d\phi} \quad \Rightarrow \quad \frac{1}{2}\phi_x^2 = U(\phi) + c$$

$$bc \Rightarrow c = 0 \qquad \frac{1}{2}\phi_x^2 = U(\phi)$$

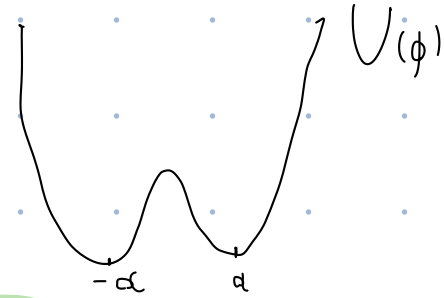
$$x - x_0 = \pm \int^{\phi} \frac{1}{\sqrt{2U(\tilde{\phi})}} \, d\tilde{\phi}$$

$$E = \int 2U \, dx$$

mass of the kink

we can boost and find a moving kink solution

Example: $U = \frac{\lambda^2}{2}\left(\phi^2 - a^2\right)^2$



$$U(\phi) \simeq \frac{1}{2}(2\lambda a)^2(\phi - a)^2 + \frac{1}{2}\frac{(2\lambda a)^2}{a}(\phi - a)^3 + \dots$$

$\Rightarrow$ $m = 2\lambda a$ mass of the scalar

pert theory: $\frac{(2\lambda a)^2}{2a} \ll 1$

$$x - x_0 = \pm \int^{\phi} \frac{1}{\sqrt{2U(\tilde{\phi})}}\, d\tilde{\phi} = \pm \frac{1}{\lambda}\int^{\phi} \frac{d\tilde{\phi}}{\tilde{\phi}^2 - a^2} = \pm \frac{1}{\lambda a}\tanh^{-1}\left(\frac{\phi}{a}\right)$$

$$\phi(x) = \pm a \cdot \tanh\left(\lambda a (x - x_0)\right)$$

$$E = \lambda^2 a^4 \int_{\mathbb{R}} \operatorname{sech}^4\left(\lambda a (x - x_0)\right) dx = \frac{4}{3}\lambda a^3$$

energy density picks at $x = x_0$

$\downarrow$

position of the kink

$$\frac{E}{m} = \frac{\frac{4}{3}\lambda a^3}{2\lambda a} = \frac{2}{3}a^2 \gg 1$$

$\uparrow$

perturbative regime

$\Rightarrow$ the kink is much more massive

Topology and Bogomolny equations

All finite energy configurations approach the vacuum at $\pm\infty$

$$\phi_\pm = \lim_{x \to \pm\infty} \phi$$

If $\phi_+ = \phi_- \Rightarrow \phi(x)$ can be continuously deformed into the zero energy vacuum $\phi = \phi_+$

If $\phi_+ \neq \phi_- \Rightarrow \phi(x)$ cannot be continuously deformed into the zero energy vacuum $\phi = \phi_+$
$\Rightarrow$ topological stability of kinks

Topological charge $\quad N = \phi_+ - \phi_- = \int_{\mathbb{R}} \phi_x \, dx$

If $U \geq 0$ we can always find $W(\phi)$

superpotential

$$U(\phi) = \frac{1}{2}\left[\frac{dW(\phi)}{d\phi}\right]^2$$

$$E = \frac{1}{2}\int_{\mathbb{R}} dx \left(\phi_t^2 + \phi_x^2 + W_\phi^2\right) = \frac{1}{2}\int_{R}\left[\phi_t^2 + (\phi_x \pm W_\phi)^2 \mp 2\phi_x W_\phi\right] =$$

$$= \underbrace{\frac{1}{2}\int_{\mathbb{R}} dx \left[\phi_t^2 + (\phi_x \pm W_\phi)^2\right]}_{\geq 0} \mp \left(W(\phi(\infty)) - W(\phi(-\infty))\right)$$

$$E \geq \left| W(\phi(\infty)) - W(\phi(-\infty))\right|$$ "Bogomolny bound"

depends only on the topological data at $\pm\infty$

Say $\phi(\infty) = \phi_2 > \phi(-\infty) = \phi_1$

Minimum energy configuration:

$\phi_t = 0$ and $E = W(\phi_1) - W(\phi_2)$

$$\Rightarrow \frac{d\phi}{dx} = \frac{dW}{d\phi}$$ "Bogomolny equation"

$\Rightarrow$ solutions are kinks

field eq is a 2nd order PDE
but special solutions arise from a first-order ODE (the Bogomolny equation)

In general the field equations are usually not integrable, but the Bogomolny equations often are integrable

Example

$$U = \frac{\lambda^2}{2}(\phi^2 - a^2)^2 \qquad W = \lambda\left(a^2\phi - \frac{1}{3}\phi^3\right)$$

Bogomolny equation: $\phi_x = \lambda(a^2 - \phi^2)$

$$E = W(a) - W(-a) = \frac{4}{3}\lambda a^3$$

Higher dimensions and a scaling argument

If $\phi(x)$ is a static solution in $D+1$ dim

$$\nabla^2 \phi = \frac{dU}{d\phi}$$

$$E(\phi) = \int_{\mathbb{R}^D} d^D x \left[\frac{1}{2} |\nabla \phi|^2 + U(\phi) \right] = E_{grad} + E_V$$

consider the one-parameter family of configurations $\phi_{(c)}(x) = \phi_{(1)}(cx)$

$$E[\phi_{(c)}] = \frac{1}{c^{D-2}} E_{grad} + \frac{1}{c^D} E_V$$

minimum of $E \Rightarrow \left. \frac{d E[\phi_{(c)}]}{dc} \right|_{c=1} = 0$

$$(D-2) E_{grad} + D E_V = 0$$

- $D=1$ $\quad E_{grad} = E_V \Rightarrow$ kinks
- $D=2$ $\quad E_V = 0 \Rightarrow$ Can lead to non-linear field theories if the target space is a manifold without a linear structure $\phi : \mathbb{R}^{2,1} \to \Sigma$
  sigma models
- $D=3$ finite energy solutions do not exist
  Adding a Skyrme term $|\nabla\phi|^4 \Rightarrow$ Skyrmions

Kinks in $D>1 \Rightarrow$ "domain walls"
infinite energy

Sigma model lumps

$$\phi: \mathbb{R} \times \mathbb{R}^2 \longrightarrow S^{N-1} \qquad \phi^a(x^\mu) \in \mathbb{R}^N$$

$$\sum_{a=1}^{N} \phi^a \phi^a = 1$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a$$

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2} \lambda(x^\mu) \left(1 - \sum \phi^a \phi^a \right)$$

$$\Box \phi^a - \lambda \phi^a = 0 \qquad \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$$

$$\Box \phi^a - \left(\phi^b \Box \phi^b \right) \phi^a = 0$$

$$\mathcal{L} = \frac{1}{2} g_{pq}(\phi) \eta_{\mu\nu} \partial^\mu \phi^p \partial^\nu \phi^q \qquad p, q, r = 1, \dots, N-1$$

$$g_{pq} = \delta_{pq} + \frac{\phi^p \phi^q}{1 - \sum \phi^r \phi^r}$$

Polyakov action: $S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$

Focus on $N=3$, time-independent solutions with finite energy $\int \mathcal{L}\, d^2x$

$\hookrightarrow$ $r\,|\nabla\phi^a| \to 0$ as $r\to\infty$

$\phi(x^i)$ ~~tends~~ tends to a constant ϕ^∞ at infinity

choose $\phi^\infty = (0,0,1)$

Laplacian in 2-d is conformally inv:

$$c\Delta_{cg} = \Delta_g \qquad (g \to c(x^i)\,g)$$

$$\Delta_g = |g|^{-\frac{1}{2}}\,\partial_i\left(|g|^{1/2} g^{ij}\partial_j\right)$$

$\Rightarrow$ $\Delta\phi^a - (\phi^b\Delta\phi^b)\phi^a = 0$ are satisfied on S^2

(since it is the compact compactification of $\mathbb{R}^2$)

conformal invariance $\Rightarrow$ spatial rescaling does not change the energy

$\Rightarrow$ shrink any static solution to zero

the solitons do not fully deserve their name and are called "lumps"

Continuous map: $\phi : S^2 \to S^2$

topological degree | $Q = \deg \phi = \frac{1}{8\pi} \int_{S^2} \varepsilon^{ij} \varepsilon^{abc} \phi^a \partial_i \phi^b \partial_j \phi^c d^2x$

partially characterizes static solutions

↳ only global info

fields can have different energies within one topological sector

$$E = \frac{1}{2}\int_{S^2} \partial_i \phi^a \partial_i \phi^a \, d^2x \geq 4\pi|\rho|$$

"="

spin waves

$$\partial_i \phi^a = \pm \varepsilon_{ij} \varepsilon^{abc} \phi^b \partial_j \phi^c$$

first-order Bogomolny equations

$$f : \mathbb{R}^{2,1} \longrightarrow \mathbb{CP}^1$$

Solutions:

$$\phi^1 + i\phi^2 = \frac{2f}{1+|f|^2} \qquad \phi^3 = \frac{|f|^2 - 1}{|f|^2 + 1}$$

The Bogomolny eq imply: f is holomorphic or anti-holomorphic in $z = x^1 + ix^2$