Frobenius thm

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FROBENIUS THM

Def. Submanifold $M \subset X$ is called integrable mfd of the bundle $E \subset TX$, if $T_x M \subset E_x \ \forall x \in M$.

Intuitively: tangent plane is therefore included in this subbundle

Def. Subbundle $E \subset TX$ is called involutive, if for any two vector fields v, w on X, that are tangent to E, their commutator $[v,w]$ is tangent to E as well.

Frobenius thm

Subbundle $E \subset TX$ is involutive iff it is integrable (integrable = for every $p \in X$ there exists integrable submanifold $p \in M \subset X$)

Also: $\dim M = \operatorname{rank} E = m$

(Non)-example (or why doesn't both of the conditions hold for every bundle)

Consider $\mathbb{R}^3$ and two fields $v = \frac{\partial}{\partial x}$ and $w = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$

Consider $E^2 \subset T\mathbb{R}^3$ generated by v and w

But: $[v,w] = -\frac{\partial}{\partial z}$ (not tangent to E!)

$\Rightarrow$ not involutive

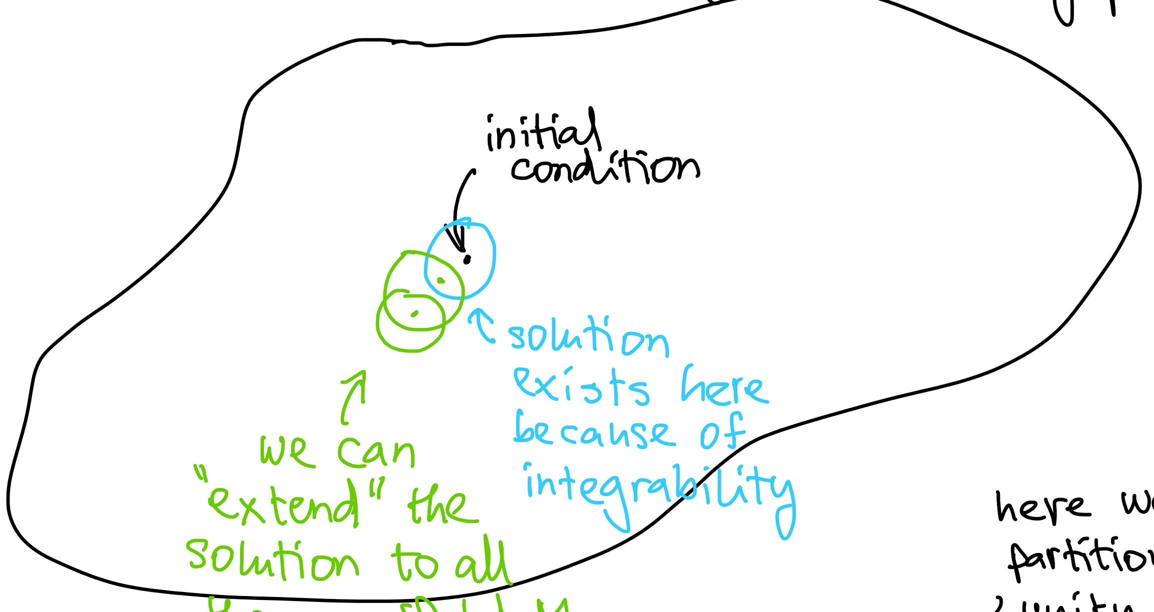
Integrable? Vector fields $v, w, [v,w]$ are the basis of tangent space in every point
$\Rightarrow$ therefore there doesn't exist integrable manifold $M^2 \subset \mathbb{R}^3$ in any point
↳ should be 2-dim!

Fed up of mathematics?

What this means concretely?

Integrability intuitively:

- we have two manifolds $M \subset X$ one inside of another; integrability condition: can we locally write field at any point of M only using coordinates of M (and not X)
- in other words: if we do a small perturb. in any point, will we still stay in M or enter $X \setminus M$ instead
- existence of solutions to differential eq.? What are we effectively doing when solving differential eq.? (Differential eq. tells us prescriptions how to go to a nearby point...)



here we used partition of

the manifold M ← duality
because we have integrability
in every point!

⇒ so the solution of system of diff. eq.
exists on the whole manifold

Involution: condition $[w,v] = f(v,w)$
· bundle's "curvature" is 0

Frobenius thm & lax pair:
- in Arianna's chapter: had pair (L_0, L_1); such that $[L_0, L_1] = 0$ (special case of above involution condition; also L_0, L_1 were matrices)
- in today's chapter: we actually see (L_0, L_1) rather as vector fields; in general:

$$[L_0, L_1] \in \mathrm{Span}\{L_0, L_1\}$$

- in practise: ansatz: $[L_0, L_1] = aL_0 + bL_1$ (scalars) and solve for a & b!
- so this is our condition for overdetermined system to be solvable!
  = ASD condition eq.

SUMMARY:
So the following things provide same data:
(i) ASD equation (eg. $F_{\alpha\beta} = 0$)
(ii) lax pair
(iii) integrability property
(iv) overdetermined system of eq. has a solution
(v) locally, fields on $M \subset X$ can be

written only using coords in ...

FT

"Appendix:"

- Lax pair (def. wiki): pair of operators $(L(t), M(t))$ that:

$$\frac{dL}{dt} = [M, L] \qquad \text{(Lax eq.)}$$

- but we can see this eq. as compatibility condition for the following system of eqns:

$$L\psi = \lambda\psi$$
$$\frac{d\psi}{dt} = M\psi$$

- in the book: $\frac{d\psi}{dt} = 0$, $\lambda = 0$: $L\psi = 0$, $M\psi = 0$

+ compatibility condition: $[M, L] = 0$

- this is equivalent to Frobenius thm; with $f(v, w) = 0$
