Recap

Def

A category $\mathcal{C}$ is
- a collection $ob(\mathcal{C})$ of objects
- $\forall M, N \in ob(\mathcal{C})$, a collection of morphisms from M to N (can be empty), sometimes written as $Hom_{\mathcal{C}}(M,N)$.

- $\forall L, M, N \in ob(\mathcal{C})$, a function $Hom_{\mathcal{C}}(M,N) \times Hom_{\mathcal{C}}(L,M) \to Hom_{\mathcal{C}}(L,N)$
$(g, f) \mapsto g \circ f$
(composition)
- $\forall M \in ob(\mathcal{C})$, an element $id_M \in Hom_{\mathcal{C}}(M,M)$ (identity)

such that
- (associativity) : $\forall f \in Hom(L,M)$, $g \in Hom(M,N)$, $h \in Hom(N,P)$ we have
$(h \circ g) \circ f = h \circ (g \circ f)$
- (identity) : $\forall f \in Hom(L,M)$, $f \circ id_L = f = id_M \circ f$.

Examples

Set, Grp, Ring, R-Mod, Mod-R, Ab

(Summary of Category Theory in appendix)

Def

Let R be a ring. A right R-module is an abelian group M equipped with a map $M \times R \to M$, $(m,r) \mapsto mr$ such that $\forall r, s \in R$, $m, n \in M$,

- $(m+n)r = mr + nr$
- $m(r+s) = mr + ms$
- $m(rs) = (mr)s$
- $m \cdot 1 = m$

Mod-R

Objects: right R-modules
Morphisms: right R-homomorphisms $(f(mr + m'r') = f(m)r + f(m')r')$

Def A functor $F : \mathcal{A} \to \mathcal{B}$ consists of
- a function $ob(\mathcal{A}) \to ob(\mathcal{B})$, $A \mapsto F(A)$
- $\forall A, A' \in \mathcal{A}$, a function $Hom_{\mathcal{A}}(A,A') \to Hom_{\mathcal{B}}(F(A), F(A'))$, $f \mapsto F(f)$

s.t.
i) $F(f' \circ f) = F(f') \circ F(f)$ $\forall A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$
ii) $F(1_A) = 1_{F(A)}$ $\forall A \in \mathcal{A}$

1.1 Complexes of R-modules

Given $A, B, C \in \text{Mod-}R$, $f: A \to B$, $g: B \to C$ (right) R-homomorphisms, we can form the sequence $A \xrightarrow{f} B \xrightarrow{g} C$.

The sequence is exact (at B) if $\ker(g) = \operatorname{im}(f)$. It follows that $g \circ f = 0$.

Def

A chain complex $C.$ of R-modules is a family $\{C_n\}_{n \in \mathbb{Z}}$ of R-modules, with R-homomorphisms $d_n: C_n \to C_{n-1}$ s.t. $d_{n-1} \circ d_n : C_n \to C_{n-2}$ is zero $\forall n$.
the "differentials" of $C.$

From this we get:
- The module of n-cycles of $C.$: $Z_n = Z_n(C.) = \ker(d_n)$.
- The module of n-boundaries of $C.$ $B_n = B_n(C.) = \operatorname{im}(d_{n+1})$

(Note Since $d \circ d = 0$ we have $0 \subseteq B_n \subseteq Z_n \subseteq C_n$ $\forall n$)
- The n^{th} homology module of $C.$ $H_n(C.) = Z_n / B_n$.

Write $C = C.$.

Def

A morphism of chain complexes $u: C. \to D.$ (i.e. a chain complex map) is a family of R-homomorphisms $u_n : C_n \to D_n$ s.t. $u_{n-1} d_n = d_n u_n$, i.e. s.t.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r, "d_{n+2}"] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r, "d_{n-1}"] & \cdots
\end{tikzcd}$$

commutes.

typo in Weibel

Category: $Ch(\text{Mod-}R)$

1.1.2

A morphism $u: C. \to D.$ of chain complexes sends boundaries to boundaries and cycles to cycles, hence maps $H_n(C.) \to H_n(D.)$.

Proof

Write Cd_n, Dd_n etc. Then by definition of u, $u_{n-1} Cd_n = Dd_n u_n$ $\forall n$.
Let $z \in Z_n(C.) = \ker(Cd_n)$. wts $u_n(z) \in Z_n(D.) = \ker(Dd_n)$
We have $Cd_n(z) = 0$, so $u_{n-1} Cd_n(z) = 0 = Dd_n u_n(z)$, i.e. $u_n(z) \in Z_n(D.)$.
Let $b \in B_n(C.) = \operatorname{im}(Cd_n)$. Then $\exists x \in C_{n+1}$ s.t. $Cd_{n+1}(x) = b$.
wts $u_n(b) \in B_n(D.)$, i.e. $\exists y \in D_{n+1}$ s.t. $Dd_{n+1}(y) = u_n(b)$.
Since $b = Cd_{n+1}(x)$, $u_n(b) = u_n Cd_{n+1}(x) = Dd_{n+1}(u_{n+1}(x))$, so take $y = u_{n+1}(x)$.
Then $u_n(b) \in B_n(D.)$

Def

A morphism $C_\cdot \to D_\cdot$ of chain complexes is called a <u>quasi-isomorphism</u> if the maps $H_n(C_\cdot) \to H_n(D_\cdot)$ are all isomorphisms

Def   Reindex $C^n = C_{-n}$

A <u>cochain complex</u> $C^\cdot$ of R-modules is a family $\{C^n\}$ of R-modules, together with maps $d^n : C^n \to C^{n+1}$ s.t. $d \circ d = 0$.

From this we get

- The module of <u>n-cocycles</u> $Z^n(C^\cdot) = \ker(d^n)$
- The module of <u>n-coboundaries</u> $B^n(C^\cdot) = \mathrm{im}(d^{n-1})$
- The n^{th} cohomology module of $C^\cdot$ $H^n(C^\cdot) = Z^n / B^n$.

Morphisms & quasi-isomorphisms are defined as for chain complexes.

Def

i) A chain complex $C_\cdot$ is called <u>bounded</u> if almost all the C_n are zero.

ii) If $C_n = 0$ unless $a \leq n \leq b$, we say that the complex has <u>amplitude</u> in $[a, b]$.

iii) A complex $C_\cdot$ is <u>bounded above (below)</u> if $\exists$ a bound $b\,(a)$ s.t. $C_n = 0$ $\forall\, n > b \; (n < a)$.

These types of chain complexes all form full subcategories of $Ch = Ch(R\text{-Mod})$
Ch_b, Ch_-, Ch_+

(Similarly for cochain complexes : Ch^b, Ch^-, Ch^+, $Ch^{\geq 0}$).

1.2 Operations on Chain Complexes   (Abelian Categories)

Def

A category $\mathcal{A}$ is called an <u>Ab-category</u> (or pre-additive category) if
category of abelian groups

- every hom-set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ forms an abelian group under addition
- composition distributes over addition: given $A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$ we have $h(g + g')f = hgf + hg'f$ in $\mathrm{Hom}(A, D)$.

Ex

Ch is an Ab-category: add chain maps degreewise - $\{f_n\} + \{g_n\} = \{f_n + g_n\}$.

Def

An <u>additive functor</u> $F : \mathcal{B} \to \mathcal{A}$ between Ab-categories is a functor s.t. each $\mathrm{Hom}_{\mathcal{B}}(B', B) \to \mathrm{Hom}_{\mathcal{A}}(F(B'), F(B))$ is a group homomorphism.
(i.e. $F(f + g) = F(f) + F(g)$).

Def

An additive category is an Ab category $\mathcal{A}$ with a zero object (i.e. an object that is initial & terminal), and $\forall A, B \in \mathcal{A}$, a product $A \times B$.

Note

This structure is enough to make finite products the same as finite coproducts.

Ex

Ch is an additive category. The zero object is the complex 0 of zero modules $\cdots \to 0 \to 0 \to 0 \to \cdots$. Given a family $\{A_\alpha\}$ of complexes of R-modules, the product $\prod_\alpha A_\alpha$ and coproduct $\bigoplus_\alpha A_\alpha$ exist in Ch and are defined degreewise: the differentials are $\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1}$ and $\oplus d_\alpha : \bigoplus_\alpha A_{\alpha,n} \to \bigoplus_\alpha A_{\alpha,n-1}$.

Rmk

$\oplus H_n(A_\alpha) \cong H_n(\oplus A_\alpha)$, $\prod H_n(A_\alpha) \cong H_n(\prod A_\alpha)$ $\forall n$.

Def

A chain complex B is called a subcomplex of C if each B_n is a submodule of C_n and the differential on B is the restriction of the differential on C.

$\Leftrightarrow$ the inclusions $i_n : B_n \hookrightarrow C_n$ constitute a chain map $B \to C$.

$\hookrightarrow$ In this case we can form the quotient complex C/B given by
$$\cdots \to C_{n+1}/B_{n+1} \xrightarrow{d} C_n/B_n \xrightarrow{d} C_{n-1}/B_{n-1} \xrightarrow{d} \cdots$$

Ex

Given a chain map $f : B \to C$, $\{\ker(f_n)\}$ (the kernels of the f_n) is a subcomplex of B denoted $\ker(f)$ and $\{\operatorname{coker}(f_n)\}$ is a quotient complex of C denoted $\operatorname{coker}(f)$.

Def Let $\mathcal{A}$ be an additive category.

i) A kernel of a morphism $f : B \to C$ is a map $i : A \to B$ s.t. $f \circ i = 0$ and for any morphism $g : X \to B$ s.t. $f \circ g = 0$, $\exists!$ morphism $\bar{g} : X \to A$ s.t. $g = i \circ \bar{g}$.

$$\begin{tikzcd} X \arrow[d, dashed, "\bar{g}"'] \arrow[dr, "g"] & & \\ A \arrow[r, "i"] & B \arrow[r, "f"] & C \end{tikzcd}$$

ii) A cokernel of $f : B \to C$ is a map $e : C \to D$ s.t. $e \circ f = 0$ and for any morphism $h : C \to Y$ s.t. $h \circ f = 0$, $\exists!$ morphism $\bar{h} : D \to Y$ s.t. $h = \bar{h} \circ e$.

$$\begin{tikzcd} & & Y \\ B \arrow[r, "f"] & C \arrow[ur, "h"] \arrow[r, "e"] & D \arrow[u, dashed, "\bar{h}"'] \end{tikzcd}$$

iii) A map $i : A \to B$ in $\mathcal{A}$ is monic if $i \circ g = 0 \Rightarrow g = 0$ $\forall g : A' \to A$.

iv) A map $e : C \to D$ is an epi if $h \circ e = 0 \Rightarrow h = 0$ $\forall h : D \to D'$.

Rmk Every kernel is monic & every cokernel is an epi.

Def An abelian category is an additive category $\mathcal{A}$ s.t.
a) every map in $\mathcal{A}$ has a kernel & a cokernel
b) every monic in $\mathcal{A}$ is the kernel of its cokernel
c) every epi in $\mathcal{A}$ is the cokernel of its kernel.

Ex Mod-R is an abelian category.

Rmk In an abelian category, the image $\mathrm{im}(f)$ of $f: B \to C$ is the subobject $\ker(\operatorname{coker} f)$ of C.

Def A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an abelian subcategory if it is abelian, and an exact sequence in $\mathcal{B}$ is also exact in $\mathcal{A}$.

For an abelian category $\mathcal{A}$, we can form $Ch(\mathcal{A})$ in the same way as $Ch(\text{Mod-}R)$ and $H_n: Ch(\mathcal{A}) \to \mathcal{A}$ is a functor.

Thm (1.2.3)
$Ch(\mathcal{A})$ is an abelian category.

Def
A double complex (or bicomplex) in $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps $d^h: C_{p,q} \to C_{p-1,q}$ and $d^v: C_{p,q} \to C_{p,q-1}$ s.t. $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$. Denote this by $C_{\cdot\cdot}$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v"'] & C_{p,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v"'] & C_{p,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d^h"'] \arrow[d] & C_{p+1,q-1} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

each square anticommutes.

We say a double complex is bounded if it has only finitely many nonzero terms along each diagonal line $p+q=n$.

Sign trick
The d^v are not maps in Ch (anticommutivity), but we can define chain maps $f_{*q}: C_{*q} \to C_{*q-1}$ by introducing $\pm$ signs: $f_{p,q} = (-1)^p d^v_{p,q} : C_{p,q} \to C_{p,q-1}$.
↳ We can identify the category of double complexes with $Ch(Ch)$.

Def

Define the total complexes $Tot(C) = Tot^{\Pi}(C)$ and $Tot^{\oplus}(C)$ by

$Tot^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q}$ and $Tot^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$.

The formula $d = d^h + d^v$ defines maps $d : Tot^{\Pi}(C)_n \to Tot^{\Pi}(C)_{n-1}$ and $d : Tot^{\oplus}(C)_n \to Tot^{\oplus}(C)_{n-1}$ s.t. $d \circ d = 0$, so these are chain complexes.

(Relevant to spectral sequences).

We say an abelian category is complete if Tot^{Π} exists, and cocomplete if $Tot^{\oplus}$ exists.

↳ Mod-R and Ch(Mod-R) are complete & cocomplete.

Translation : $C[p]_n = C_{n+p}$, $C[p]^n = C^{n-p}$, with differential $(-1)^p d$.

↳ $H_n(C[p]) = H_{n+p}(C)$, $H^n(C[p]) = H^{n-p}(C)$.

↳ We can make translation a functor by defining $f[p]_n = f_{n+p}$ (or $f[p]^n = f^{n-p}$) for a chain map $f : C \to D$.

1.3 Long exact sequences

Thm 1.3.1

Let $0 \to A. \xrightarrow{f} B. \xrightarrow{g} C. \to 0$ be a ses of chain complexes. Then $\exists$ natural maps $\partial : H_n(C) \to H_{n-1}(A)$ (called connecting homomorphisms), s.t.

$\dots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \dots$ is an exact sequence.

$z \mapsto f_{n-1}^{-1} d_n g_n^{-1}(z)$

(Similarly for cochain complexes).

Ex 1.3.1

Let $0 \to A \to B \to C \to 0$ be a ses of complexes. Show that if 2 of the 3 complexes A, B, C are exact (i.e. exact at each A_n (for $A.$)), then so is the third.

↳ i.e. $im(d_i) = ker(d_{i-1})$ $\forall i$

3x3 lemma

Consider the following diagram in an abelian category

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that every column is exact.

Then

i) if the bottom two rows are exact, so is the top row.

ii) if the top two rows are exact, so is the bottom row.

iii) if the top and bottom rows are exact and the composite $A \to C$ is zero, the middle row is also exact.

Snake lemma (Proof given in "It's my Turn", 1980)

Consider a commutative diagram of R-modules of the form

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C & \end{tikzcd}$$

If the rows are exact then there is an exact sequence

$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$

with ∂ defined by $\partial(c') = i^{-1} g p^{-1}(c')$ $(c' \in \ker(h) \subseteq C')$.

Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$, and if $B \to C$ is surjective, then so is $\operatorname{coker}(g) \to \operatorname{coker}(h)$.

Rmk

The snake lemma holds in any abelian category.

5-lemma

In any commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "a"', "\cong"] & B' \arrow[r] \arrow[d, "b"', "\cong"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E \end{tikzcd}$$

with exact rows in any abelian category, if a, b, d, e are isomorphisms, then c is also an isomorphism. More precisely, if b & d are monic & a is an epi, then c is monic and dually if b & d are epis and e is monic, then c is an epi.

Constructing ∂

Consider the ses of chain complexes $0 \to A \to B \to C \to 0$. From the snake lemma and the diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Z_n A \arrow[r] \arrow[d] & Z_n B \arrow[r] \arrow[d] & Z_n C \arrow[d] & \\ 0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\ & \frac{A_{n-1}}{dA_n} \arrow[r] \arrow[d] & \frac{B_{n-1}}{dB_n} \arrow[r] \arrow[d] & \frac{C_{n-1}}{dC_n} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

We see that the rows are exact in the diagram

$$\begin{tikzcd} & \frac{A_n}{dA_{n+1}} \arrow[r] \arrow[d, "d"] & \frac{B_n}{dB_{n+1}} \arrow[r] \arrow[d, "d"] & \frac{C_n}{dC_{n+1}} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f"] & Z_{n-1}(B) \arrow[r, "g"] & Z_{n-1}(C). & \end{tikzcd}$$

$\ker\left(d : \frac{A_n}{dA_{n+1}} \to Z_{n-1}(A)\right) = H_n(A)$ and $\operatorname{coker}\left(d : \frac{A_n}{dA_{n+1}} \to Z_{n-1}(A)\right) = H_{n-1}(A)$.

Therefore the snake lemma gives us the exact sequence

$H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$.

1.4 Chain Homotopies

Def

A complex C is called split if there are maps $s_n : C_n \to C_{n+1}$ s.t. $d = dsd$.

The s_n are called splitting maps.

If in addition $H_n(C_\cdot) = 0$ $\forall n$ (i.e. C is acyclic), then we say C is split exact.

Def

Let $f, g : C \to D$ be chain maps. We say f and g are chain homotopic if $\exists$ maps $s_n : C_n \to D_{n+1}$ s.t. $f_n - g_n = s_{n-1} d_n + d_{n+1} s_n$ $\forall n$.

The $\{s_n\}$ are called a chain homotopy from f to g.

We say $f : C \to D$ is a chain homotopy equivalence if $\exists g : D \to C$ s.t. gf and fg are chain homotopic to the identity.

Lem

If f and g are chain homotopic then they induce the same maps $H_n(C) \to H_n(D)$.