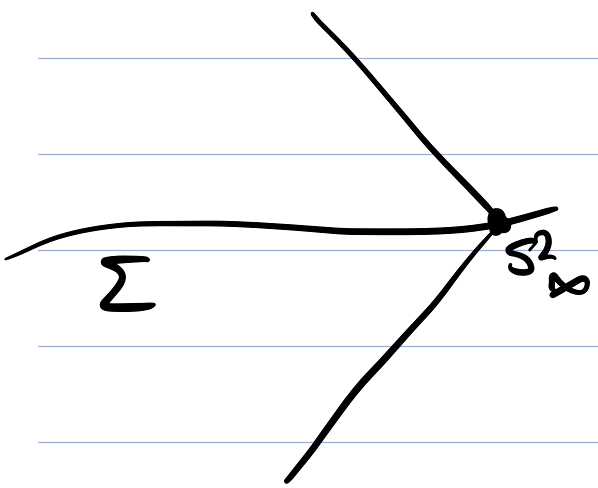
Gravitational Instantons

Refs: Dunajski Book, Dunajski 25' GI Old & New, Hawking 77.

Aside: No Gravitational solitons.



$$M_{\text{Komar}} \sim \int_{S^2_\infty} * \, dk = 0$$

$$= \int_\Sigma d * dk = 0$$

So Positive mass theorem $\Rightarrow$ Minkowski.

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Motivation:

YM Instantons = finite action soln to Euclidean field eqns describing tunnelling between vacua.

$$Z = \int \mathcal{D}g \, e^{-I_E[g]} \simeq \sum e^{-I_E[\text{Instantons}]}$$

Grav instantons (Hawking 77') have the same Goal.

Sol to 4d Einstein (Maxwell) $(+ + + +)$

Complete, asymptotically 'flat' and
$\int |Rm|^2 < \infty$ (finite energy)

- Most are not analytic continuations of Lorentzian, However ...

Why? Physics ...

→ Predict BH thermodynamics.

→ GI dominate Euclidean path integral

→ so if Quantum Gravity reduces to GR in classical limit then EQG will be similar to WKB in QM → Newtonian

Why? Maths

→ Black hole uniqueness theorem

→ Kähler / Hyper kähler Geometry.

Examples :

① Schwarzschild. $t \to i\tau$

$$\underbrace{\left(1-\frac{2M}{r}\right)^{-1}dr^2}_{d\rho^2} + \left(1-\frac{2M}{r}\right)d\tau^2 + r^2\, d\Omega^2_{S^2}.$$

(a) restrict $r \geq 2M$

(b) Make τ periodic:

we have KVF $\partial_\tau, \partial_\phi$,

$$\|\partial_\tau\|^2 = \left(1-\frac{2M}{r}\right) \qquad \|\partial_\phi\|^2 = r^2\sin^2\theta$$

When $r \to 2M$ issues

$\rho = 4M\sqrt{1-\frac{2M}{r}}$ approach $r=2M$ from above

$$g \sim \underbrace{d\rho^2 + \frac{\rho^2 d\tau^2}{16M^2}} + 4M^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$\downarrow$

regular provided $\tau \sim \tau + \underbrace{8\pi M}_{\beta}$.

Next: left invariant one-forms on $SU(2)$.

$$\sigma_1 + i\sigma_2 = e^{i\psi}(d\theta + i\sin\theta\, d\phi)$$
$$\sigma_3 = d\psi + \cos\theta\, d\phi$$

$$0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi \quad 0 \leq \psi \leq 4\pi.$$

$$g_{\mathbb{R}^4} = d\rho^2 + \frac{1}{4}\rho^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

Ex: Eguchi Hanson

$$g = \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{1}{4}r^2\left(1 - \frac{a^4}{r^4}\right)\sigma_3^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2)$$

set $\rho^2 = r^2(1 - (a/r)^4)$

near $r=a$ $\quad g \sim \frac{1}{4}(d\rho^2 + \rho^2 d\psi^2)$

Smooth provided ψ period 2π (not 4π)

Hence as $r \to \infty$

$$g \longrightarrow d\rho^2 + \frac{1}{4}\rho^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

$$0 \le \theta \le \pi \qquad 0 \le \phi \le 2\pi \qquad 0 \le \psi \le 2\pi .$$

i.e. the metric on $\mathbb{R}^4 / \Gamma \qquad \Gamma = \mathbb{Z}_2 .$

this is <u>ALE</u> asymptotics (Γ finite subgp $SU(2)$)

Kronheimer $\rightarrow$ $\exists!$ Hyperkahler ALE GI for each Γ.

<u>Ex</u>: Taub NUT

$$g = \frac{1}{4}\frac{r+m}{r-m} dr^2 + m^2 \frac{r-m}{r+m}\sigma_3^2 + \frac{1}{4}(r^2 - m^2)(\sigma_1^2 + \sigma_2^2)$$

$r = m \longrightarrow$ NUT singularity (later)

set $r = m + \rho^2/2m \qquad g \sim d\rho^2 + \frac{\rho^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

$\therefore$ Smooth extension.

Note this approaches an S^1 bundle over S^2 as $r \to \infty$.

<u>Def</u>: ALF if approaches S^1 bundle over S^2 at infinity. AF

<u>Nuts & Bolts:</u> (Gibbons, Hawking 79)

$G \ni \mu_\tau : M \to M$ $\quad \tau$ parameter.

$K = K^a \dfrac{\partial}{\partial x^a} = \dfrac{\partial}{\partial \tau}$ the Killing field

$K = 0 \to$ fixed point

At this fixed point $\mu_{\tau *} : T_pM \to T_pM$

$\mu_{\tau *}$ generated by the antisymmetric matrix $K_{a;b}$, this can have rank 0, 2 or 4.

Rank 0 $\Rightarrow$ the KVF is zero everywhere, G trivial.

$\Rightarrow$ rank 2 $\longrightarrow$ Bolt singularity
rank 4 $\longrightarrow$ Nut singularity.

these give <u>topological</u> invariants!

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ASD in Riemannian Geometry:

<u>def</u>: g, ∇, $\Gamma^{\mu}_{\nu\rho}$, $R_{\mu\nu\gamma\delta}$, $R_{\mu\nu}$, R

<u>Ex</u>: Moving frame.

$$g = \delta_{ab}\, e^a \otimes e^b \quad , \quad e^a = e^a{}_\mu\, dx^\mu$$

<u>Cartan</u>:
$$de^a + \omega^a{}_b \wedge e^b = 0$$
$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$
$$= \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$$

the volume elt $\quad \varepsilon = e^1 \wedge \dots \wedge e^n = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n$

Hodge $*$ is

$$*(e^{a_1}\wedge\dots\wedge e^{a_p}) = \frac{1}{(n-p)!}\,\varepsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_n}\, e^{a_{p+1}}\wedge\dots\wedge e^{a_n}$$

for $n=4$, on two forms $*^2 = \mathrm{Id}$

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

$F\in\Omega^2(M)$ then $F_\pm = \frac{1}{2}(F \pm *F) \in \Lambda^2_\pm$

think of $R:\Lambda^2 \longrightarrow \Lambda^2$ then

$$R = \begin{array}{c} \\ + \\ - \end{array}\underbrace{\begin{array}{c|c} + & - \\ \left(\begin{array}{c|c} C_+ & 0 \\ \hline 0 & C_- \end{array}\right) & \end{array}}_{\text{Weyl tensor}} + \underbrace{\frac{R}{12}\mathbb{1}}_{\text{pure trace}} + \underbrace{\begin{pmatrix} 0 & \Phi \\ \Phi & 0 \end{pmatrix}}_{\text{trace free}}$$

Conf. inv. trace free.

Note Ricci flat + $C_+ = 0 \Rightarrow R$ self dual

$$R_{abcd} = -\frac{1}{2}\varepsilon_{ab}{}^{pq} R_{pqcd}.$$

the taub NUT & Eguchi-Hansen are both ASD in this sense.

Kähler & HyperKähler metrics.

→ Note can be done with Spin dyad, we will be more pedestrian.

A complex manifold is a manifold with a holomorphic atlas i.e. $\phi_{uv} : \Omega \to \mathbb{C}^m$ is holomorphic.

Note: $z_a = x_a + i y_a$ then

$$i : \frac{\partial}{\partial x^a} \longmapsto \frac{\partial}{\partial y^a}$$

$$\frac{\partial}{\partial y^a} \longmapsto - \frac{\partial}{\partial x^a}$$

Induces $I : TM \to TM$ with $I^2 = -\mathrm{Id}$.

A complex structure.

if M is real manifold.

An ACS is $I: TM \to TM$, $I^2 = -\mathrm{Id}$.

write $$TM_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} TM$$
$$= \underset{\text{"}\partial_z\text{"}}{T^{1,0}M} \oplus \underset{\text{"}\partial_{\bar{z}}\text{"}}{T^{0,1}M}$$

$$\Omega^1(M)_{\mathbb{C}} = \underset{\text{"}dz\text{"}}{\Omega^{1,0}(M)} \oplus \underset{\text{"}d\bar{z}\text{"}}{\Omega^{0,1}(M)}$$

Thm (Newlander-Nirenberg) let I be an ACS then TFAE

(i) I is a CS.

(ii) $T^{0,1}M$ is integrable in the sense that $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$

(iii) $T^{1,0}M$ is integrable

(iv) $N^I = 0$

$\hookrightarrow [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$

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$g(X,Y) = g(IX, IY) \leftrightarrow$ Hermitian.

fund 2-form $g(X, IY) = \Omega(X, Y)$

def: Kähler if $\begin{cases} N^I = 0 \\ d\Omega = 0. \end{cases}$

def hyper Kähler if Kahler wrt I, J, K s.t. $IJ = K$, $JK = I$, $KI = J$

Thm: ASD + Ricci flat $\Leftrightarrow$ hyper Kähler.

Corr: Ω_j basis g SD 2-forms then if $d\Omega_j = 0$, we have $g = (e^1)^2 + \cdots + (e^4)^2$ is hyper Kähler

Reason for all of this?

Gibbons-Hawking Ansatz.

Ansatz: Harmonic function on $\mathbb{R}^3$.

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(d\tau + A)^2$$

function ⟵ indep^t of τ ⟶ 1-form.

- Tetrad $e^i = \sqrt{V}\, dx^i$ $\quad e^4 = \frac{1}{\sqrt{V}}(d\tau + A)$

  1, 2, 3.

- The SD 2 forms are

  3d hodge $*$.

  $$\Omega_i = -(d\tau + A) \wedge dx_i + \frac{1}{2} V \varepsilon_{ijk}\, dx_j \wedge dx_k$$

- Impose ASD condition $d\Omega_i = 0$

$$dA \wedge dx_i = \frac{1}{2}\, dV \wedge (\varepsilon_{ijk}\, dx_j \wedge dx_k)$$

$\Rightarrow \quad *_3 \, dV = dA .$

$\Rightarrow$ V is harmonic on $\mathbb{R}^3$ base!

Note $K = \partial_\tau$ is triholomorphic

$$dx_i = K \lrcorner \Omega_i$$

$$\Rightarrow L_K \Omega_i = d(\cancel{K \lrcorner \Omega_i}) + \cancel{K \lrcorner d\Omega_i} = 0$$

<u>Thm</u>: hyperKähler + triholom KVF
$\Rightarrow$ local Gibbons hawking.

Very nice!

Examples

Flat space:

- $V = 1/r$
- $*_3 d(1/r) = dA \Rightarrow A = \cos\theta \, d\phi$

This motivates

$$V = V_0 + \sum_{m=1}^{N} \frac{1}{|x - x_m|}$$

$$A = \sum_{m=1}^{N} \cos(\theta_m) \, d\phi_m$$

polars centred at x_m.

We get:

- $V_0 = 0$ $N=1$ flat metric
- $V_0 = 0$ $N=2$ Eguchi-Hanson
- $V_0 = 0$ $N>2$ A_{N-1} ALE (Kronheimer)
- $V_0 \neq 0$ $N=1$ Taub NUT.

Further Ideas:

- Full Classification of noncompact hyper Kähler.

- KK Monopoles

$$ds^2_5 = -dt^2 + ds^2_{\text{instanton}}$$

static, nonsingular = Soliton

- toric instanton