Amenable Groups

Rodrigo Roemig

Amenable groups admit approximately $10^{10^{10}}$ different characterizations; our goal in this ~~section~~ talk is to present a few that can be proved without too much effort. Brown-Ozawa

Rmk: G discrete group.

Def G is amenable if there exists a finitely additive left-invariant probability measure on $P(G)$.

$\mu : P(G) \to [0,\infty)$ $\qquad \mu(G)=1$

Fin. add. $\qquad \mu(A \sqcup B) = \mu(A) + \mu(B)$

$\mu(gA) = \mu(A) \quad \forall A \subseteq G \; ; \; gA = \{gh : h \in G\}$

Ex: G finite group. $\rightsquigarrow \mu(A) := \frac{|A|}{|G|}$.

Rmk: Riesz-Markov-Kakutani rep. thm

positive linear functional $\longleftrightarrow$ pos. Borel measure

$(\leftarrow)$ if we have a measure μ.

given $f \in \ell^\infty(G)$, i.e., $f : G \to \mathbb{C}$ bounded.

$\varphi(f) := \int f \, d\mu \qquad$ e.g. $f = \chi_A \Rightarrow \varphi(f) = \int \chi_A d\mu = \mu(A)$.

Def. G has a left-invariant mean if $\exists\, \varphi : \ell^\infty(G) \to \mathbb{C}$ positive linear functional with $\varphi(1) = 1 = \|\varphi\|$ and φ left-invariant.

$\hookrightarrow \varphi(g \cdot f) = \varphi(f)$, $f : G \to \mathbb{C}$, $g \cdot f(h) := f(g^{-1}h)$

Prop. G is amenable iff G has an inv. mean.

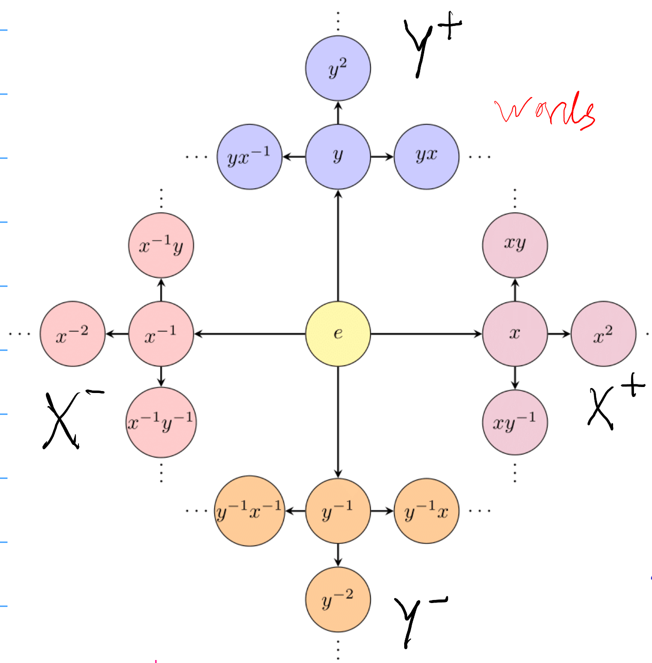
$$\varphi(f) = \int f \, d\mu \quad \longleftrightarrow \quad \mu(A) := \varphi(\chi_A).$$

Def G is paradoxical if

- $\exists\, A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$ pairwise disjoint
- $\exists\, a_1, \dots, a_n, b_1, \dots, b_m \in G$ st.

$$G = \bigsqcup_{i=1}^{n} a_i A_i = \bigsqcup_{j=1}^{m} b_j B_j .$$

Ex: $\mathbb{F}_2$ is paradoxical ($\Rightarrow$ non-amenable).



$Z = \{e, y, y^2, \dots\}$

$\mathbb{F}_2 = X^+ \sqcup X^- \sqcup (Y^+ \setminus Z) \sqcup (Y^- \cup Z)$

$\mathbb{F}_2 = X^+ \sqcup x X^- = y^{-1}(Y^+ \setminus Z) \sqcup (Y^- \cup Z)$

If $\exists\, \mu : \mathcal{P}(G) \to [0,1]$

$1 = \mu(\mathbb{F}_2) = 2\mu(\mathbb{F}_2) = 2 \quad \lightning$

$\therefore\ \mathbb{F}_2$ is non-amenable

Prop. G amenable iff G non-paradoxical.

Thm. The class of amenable groups is "nice", i.e. Amenability is preserved by...

(i) $H \leq G$, G amenable $\Rightarrow$ H amenable. subgroups

(ii) $N \trianglelefteq G$, G amenable $\Rightarrow$ G/N amenable quotients

(iii) $N \trianglelefteq G$, N and G/N amenable $\Rightarrow$ G amenable. extensions.

(iv) A direct union of amenable groups is amenable.
$G = \bigcup_{i \in I} G_i$ $\forall i, j \in I$ (poset) $\exists k \in I$ $\begin{matrix} i \leq k \\ j \leq k \end{matrix}$

Cor. G amenable iff all f.g. subgroups are amenable.

Rmk: (G, S) f.g. group.
$\beta_S(n) := \#\{g \in G : d_S(g,e) \leq n\}$ growth function.

Def. G f.g. group has subexponential growth if

$$\lim_{n\to\infty} \beta_S(n)^{1/n} = 1 \quad \left[\text{i.e., } \forall b > 1 \ \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0 \Rightarrow \beta_S(n) < b^n. \right]$$

+ does not dep on S.

Prop. G has subexp. growth $\Rightarrow$ G amenable.

Idea: Proof uses Følner sequences
$\forall n \ \exists F_n \subseteq G$ fin. s.t. $\frac{|gF_n \cap F_n|}{|F_n|} \longrightarrow 1$
$F_n = B_{d_S}(e, n)$

Prop G abelian $\Rightarrow$ G has subexp growth ($\Rightarrow$ amenable).

Proof. Suppose $G = \langle g_1, \dots, g_m \rangle$

Notice $B_{d_S}(e; n) = \{ g_1^{k_1} \cdots g_m^{k_m} : |k_1| + \dots + |k_m| \leq n \}$

$\underbrace{k_i \in [-n, n]}_{2n+1 \text{ choices}} \Longrightarrow \beta_S(n) \leq (2n+1)^m$ ∎

Cor. G solvable $\Longrightarrow$ G amenable.

↳ $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_n = G$ st. G_j / G_{j-1} is abelian.

Prop If $\exists H \leq G$ st. $H \cong \mathbb{F}_2$, then G is non-amenable.

Rmk Define

EG = class of groups constructed from finite and abelian groups by taking subgroups, quotients, extensions and directed unions.

AG = class of all amenable groups.

NF = class of groups that does not contain $\mathbb{F}_2$ as a subgroup.

Then, $$EG \underset{\substack{\downarrow \\ \text{Grigorchuk}}}{\subsetneq} AG \underset{\substack{\downarrow \\ \text{Ol'shanskii ('80)} \\ \text{Adian \& Novikov ('68/'79)}}}{\subsetneq} NF$$

"Easy counter examples"

$$H = \langle a, b, c, d, t : a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = (adacac)^4 = e,$$
$$t^{-1}at = aca, t^{-1}bt = d, t^{-1}ct = b, t^{-1}dt = c\rangle.$$

$H \in AG \setminus EG$ (Grigorchuk '96)

Theorem 14.38. *Let G_{LM} be the group generated by the following three homeomorphisms of the real line:*

$$a(t) = t+1; \quad b(t) = \begin{cases} t & \text{if } t \leq 0, \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1, \\ t+1 & \text{if } 1 \leq t; \end{cases} \quad c(t) = \begin{cases} \frac{2t}{1+t} & \text{if } 0 \leq t \leq 1, \\ t & \text{otherwise.} \end{cases}$$

Then G_{LM} is nonamenable, has no free subgroup of rank 2, is torsion-free, and has the finite presentation given by setting the following nine words in $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}$ to the identity; where $[\cdot]$ denotes the commutator:

$$\begin{array}{lll} [ba^{-1}, a^{-1}ba], & [ba^{-1}, a^{-2}ba^2], & [c, a^2b^{-1}a^{-1}], \\ [c, ab^2a^{-1}b^{-1}ab^{-1}a^{-1}], & [c, a^{-1}ba], & [c, a^{-2}ba^2], \\ [c, aca^{-1}], & [c, a^2ca^{-2}], & \\ \multicolumn{3}{l}{c^{-1}b[b, a^{-1}]cb^{-2}ab^{-1}c^{-1}b[a^{-1}, b]ab^{-1}cba^{-1}ba^{-1}.} \end{array}$$

$G_{LM} \in NF \setminus AG$ (Lodha & Moore '13)

Supramenable Groups

$(A \neq \emptyset)$

Def. G is supramenable if, for every $A \subseteq G$, there exists a fin. additive, left-invariant measure $\mu : \mathcal{P}(G) \to [0,\infty)$ with $\mu(A) = 1$.

Rmk: class of supramenable groups SAG.

Prop If G contains a free subsemigroup generated by 2 elements, then G is not supramenable.

Pf: $S = \langle a, b \rangle^+ \subseteq G$

If $\exists\ \mu : \mathcal{P}(G) \to [0,\infty)$ (properties) $\cdots\cdots$ s.t. $\mu(S) = 1$, then

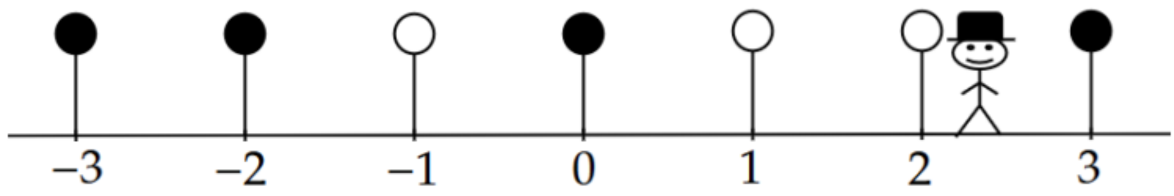
$$1 = \mu(S) = \mu(aS \sqcup bS) = \mu(aS) + \mu(bS) = 2.$$

Rmk: $SAG \subsetneq AG$

Ex Lamplighter group

$$G = \langle a, t \mid a^2 = e,\ [t^m a t^{-m}, t^n a t^{-n}] = e,\ m, n \in \mathbb{Z} \rangle$$

Figure 4 – The lamps after an action of $atat^2at^{-1}$.



$S = \langle t, at \rangle^+ \subseteq G$ is a free subsemigroup.

$\hookrightarrow$ supramenability is not preserved by extensions.

Prop G has subexponential growth $\Rightarrow$ G supramenable.

Rmk. Finite and abelian groups are supramenable.

Going even further.

* Operator algebras & Analysis

① G amenable $\Longleftrightarrow$ $C^*_r(G)$ is nuclear

② $\exists$ definition of a "amenable action" $G \curvearrowright X$ top. sp.

↳ Brown-Ozawa C^*-alg & fin-dim. app.

③ What if G is a top. group?

↳ Paterson, Amenability

* Banach-Tarski Paradox

↳ Tomkowicz, Wagon, The Banach-Tarski Paradox

* Property (T)

↳ Bekka, la Harpe, Valette. Prop. (T)

Prop. G discrete group TFAE.

① G is finite

② G is amenable and has Property (T).