Hyperbolic Coxeter Groups

≡ Coxeter Groups:

m_{ij} a "Coxeter matrix" if

- $m_{ij} \in \{2,3,\dots\} \cup \{\infty\}$
- $m_{ij} = m_{ji}$
- $m_{ii} = 1$

$\longleftrightarrow$ Graph Γ with edges

$$\underset{s_i}{\bullet} \overset{m_{ij}}{\rule{2em}{0.4pt}} \underset{s_j}{\bullet}$$

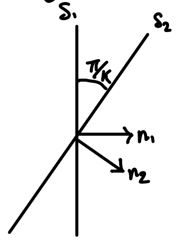
- Given a Coxeter graph Γ corr. to matrix m_{ij} define "Coxeter group"

$$W_\Gamma := \langle \{s_1, \dots, s_n\} \mid (s_i s_j)^{m_{ij}} = 1 \ \forall m_{ij} \neq \infty \rangle$$

e.g. $\Gamma = \bullet \overset{k}{\rule{4em}{0.4pt}} \bullet \implies W_\Gamma \cong$ Dihedral group order $2k$.

- The abstract definition above has a geometric interpretation.

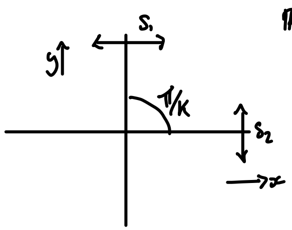
Eg. for the example above, this corresponds to the group generated by 2 reflections in $\mathbb{R}^2$ with acute angle π/K.



- In $\mathbb{E}^2$ reflection through a plane orthogonal to n is defined using the Euclidean inner product.
The properties emerging from the particular angle π/K come from the angle as defined by the Euclidean inner product.

- The reflections could instead be w.r.t. to the two coordinate axes & we could "force" the angle between these to be π/K by changing the inner product.

$\mathbb{R}^2 = \begin{pmatrix} x \\ y \end{pmatrix}$



N.B. $S_1 \rightsquigarrow n_1 \rightsquigarrow x$ coord
$S_2 \rightsquigarrow n_2 \rightsquigarrow y$ coord

- In this case, the bilinear form defining the inner product is defined by

$$\boxed{B(x_i, x_j) := -\cos(\pi/M_{ij})}$$

$B\left(\begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}\right) = B\left(\begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}0\\1\end{smallmatrix}\right) = 1$

e.g. $B\left(\begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}0\\1\end{smallmatrix}\right) = B\left(\begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}\right) = -\cos(\pi/K)$.

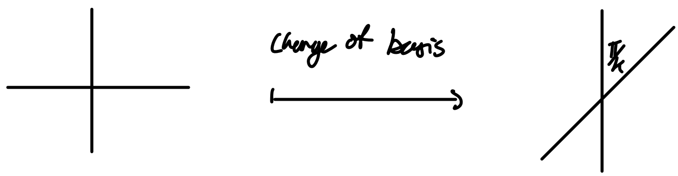
- $B : V \times V \to V \quad \leftrightsquigarrow \quad \underline{x}, \underline{y} \longmapsto \underline{x}^T B \underline{y}$

  And we can diagonalise the matrix B so that it has form

$$n\left\{\begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{n_1} & & & 0 \\ & \overbrace{-1 \ \ddots \ -1}^{n_{-1}} & & \\ 0 & & & \overbrace{0 \ \ddots \ 0}^{n_0} \end{pmatrix}\right.$$

, in which case we say B has signature (n_1, n_{-1}, n_0).

- For the $\bullet \overset{k}{\text{———}} \bullet$ example, the change of basis which gets B into its diagonal form would twist the coordinate axes to have Euclidean angle π/k.



≡ We care about the case where the signature of B is

$$(n-1, 1, 0) \rightsquigarrow \left.\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & 1 & \\ & & & -1 \end{pmatrix}\right\} n$$

i.e. "hyperbolic signature".

- In this case, consider the 2-sheeted hyperboloid.

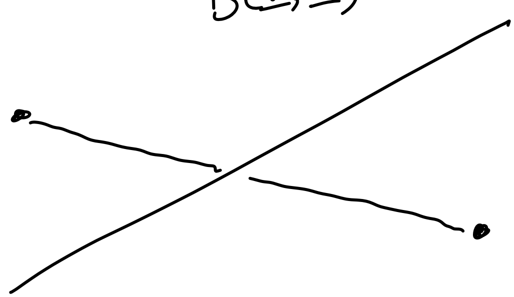
$$x_1^2 + \dots + x_{n-1}^2 - x_n^2 = -1.$$

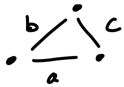
- Choosing a sheet S (e.g. $x_n > 0$), and putting a metric on S provides a model for $\mathbb{H}^{n-1}$
- Isometries of $\mathbb{H}^{n-1}$ are exactly isometries of $\mathbb{R}^n$ (wrt B) which preserve S (as a set).
- In the case of reflections (wrt B) this is exactly when the plane of reflection intersects S, and we can show this occurs for the defining reflections for W_Γ. (GOTO hyperboloid).

$\Longrightarrow W_\Gamma \hookrightarrow \mathrm{Isom}(\mathbb{H}^{n-1})$ generated by n reflections.

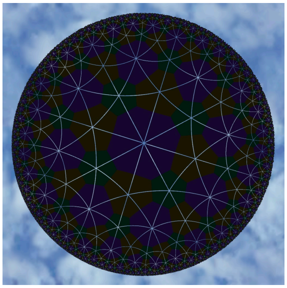
Reflections (generalised).

$$\underline{x} \longmapsto \underline{x} - 2\frac{B(\underline{x}, \underline{v})}{B(\underline{v}, \underline{v})} \circ \underline{v}$$

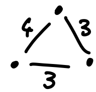


≡ In the case $n=3$, these correspond to all rank 3 Coxeter groups  s.t. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$

e.g.

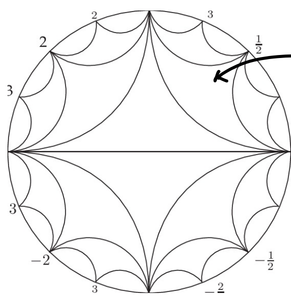


$\rightsquigarrow$



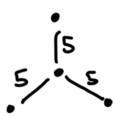
• In $\mathbb{H}^2$, the vertices of a triangle are the "extremities", so the fundamental domain of W_Γ is compact $\iff \infty \notin \{a,b,c\}$.

c.f. Farey tessellation

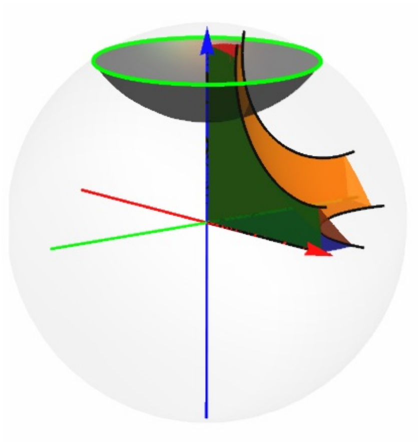


non compact.

≡ However, if W_Γ has rank 4 $\Rightarrow W_\Gamma \curvearrowright \mathbb{H}^3$, the story is quite different.

GOTO visualisation of  .

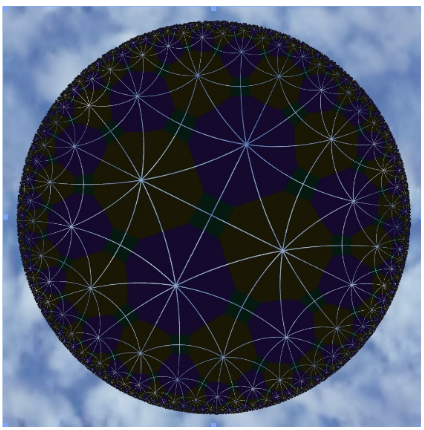
• We see that we can have well-defined inter-planar angles, but have the vertices "sticking out" beyond infinity. So, the fundamental domain is non-compact but not an ideal simplex. Since simplices are defined up to isometry by these angles (for any dimension), this is not just a feature of how we drew this simplex.



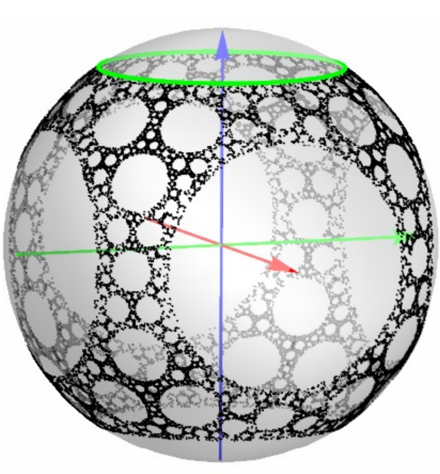
- The fundamental chamber for

$$\begin{array}{ccc} & \bullet & \\ & \mid 5 & \\ & \bullet & \\ {}^{5}\swarrow & & \searrow^{5} \\ \bullet & & \bullet \end{array}$$

extends beyond ∂H^3

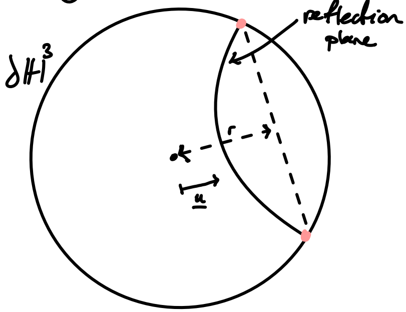


- It makes a $\bullet \overset{5}{—} \bullet \overset{5}{—} \bullet$ triangular tiling on a plane perpendicular to 3 of the planes defining the fund. chamber.



- This makes the accumulation points of the group (on ∂H^3) repel away from the disks made by this plane on ∂H^3.

- Note: For any rank 4 hyperbolic signature Coxeter group, we can describe its reflections (almost uniquely) by describing the circle that the corr. plane makes with the boundary sphere.



- To describe the arrangement of planes corr. to the Coxeter group, we need to be able to reflect planes through other planes, which corr. to conjugation in the Coxeter group.
  - This is possible, but complicated. The simplest expression I could derive is the following:

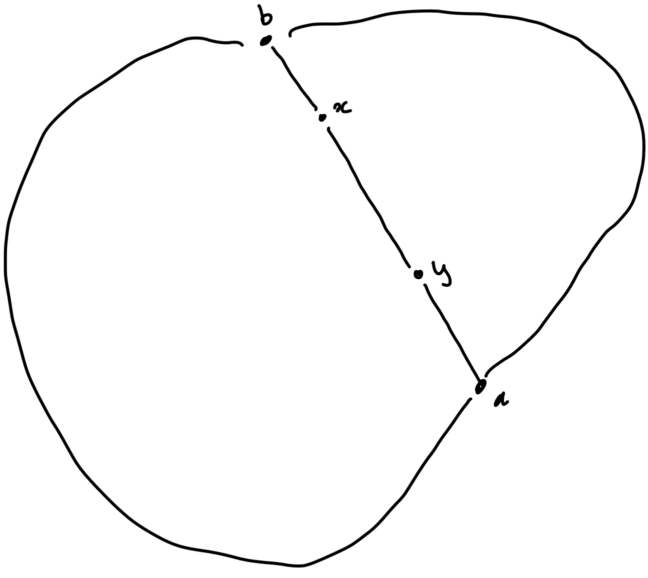
$$
\begin{aligned}
R_{p_2}(p_1) = \Bigg(& \mathrm{Sign}\Big(r(1+s^2) - 2s(\mathbf{n_1}\cdot\mathbf{n_2})\Big)\ \mathrm{Normalise}\Big[\\
& 2ars + (1-s^2)x - 2(a^2x + aby + acz), \\
& 2brs + (1-s^2)y - 2(b^2y + abx + bcz), \\
& 2crs + (1-s^2)z - 2(c^2z + acx + bcy)\ \Big], \\
& \sqrt{\frac{r^2(1+s^2)^2 - 4rs(1+s^2)(\mathbf{n_1}\cdot\mathbf{n_2}) + 4s^2((-1+c^2)(-1+y^2) + 2acxz + (-1+2c^2)z^2 + 2by(ax+cz) + b^2(-1+2y^2+z^2))}{1+s^4 - 4rs(\mathbf{n_1}\cdot\mathbf{n_2}) - 4rs^3(\mathbf{n_1}\cdot\mathbf{n_2}) + s^2(2+4r^2-4y^2+8acxz-4z^2+8by(ax+cz)+4b^2(-1+2y^2+z^2)+4c^2(-1+y^2+2z^2))}}\ \Bigg).
\end{aligned}
$$

- By work of McMullen '02, one can also show that if we naively extend the Klein tetrahedrons beyond the boundary sphere, we get a different space (with a different metric).

  - The Coxeter group acts by isometries on this space too.

  - The metric is the so-called Hibert metric, defined for any convex $X \subseteq \mathbb{R}^n$

$$d(x,y) = \log\left(\frac{|y-b|\cdot|x-a|}{|x-b|\cdot|y-a|}\right)$$



  - The orbit of all the simplices extend beyond the cone on which the hyperbola model for $\mathbb{H}^3$ exists. The new space and metric mentioned above incorporates these regions beyond the cone.