Rafaela: Generalisations of Hyperbolicity 23/3/2026

⤳ In the GGT lectures, we saw the definition of hyperbolic space, and how that can pass onto a group, given a "nice" action of the group onto the space

proper cocompact

⤳ It turns out, a lot of groups have this property! ⤳ this leads to a lot of properties that we saw in lectures.

⤳ But, life exists outside of this ~ even when a group is not hyperbolic, it can have some "hyperbolic-like" structures/properties. Relaxing the assumptions for the space X, or the "nice" group action on it gives generalisations of hyperbolicity, and allows us to tell whether groups are "kinda" hyperbolic.

Today's Plan:

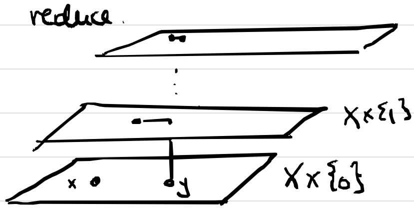
1) Relative hyperbolicity (Quick overview)
2) Hyperbolically embedded subgroups
3) Types of actions of gps on spaces
4) Acylindrical Actions
5) Acylindrically Hyperbolic groups

1) Relatively Hyperbolic Groups:

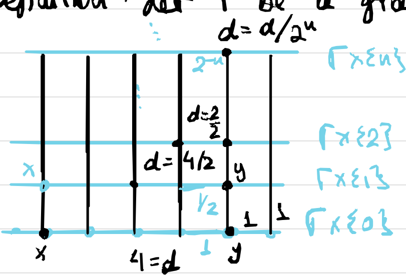
The combinatorial horoball: $\mathcal{H}$.

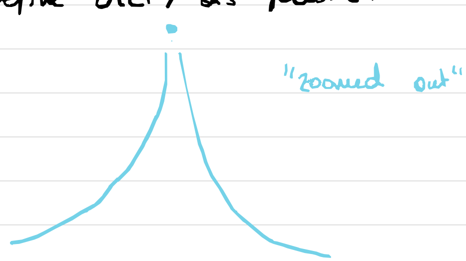
Idea: Given a space X, take a bunch of copies of it. As you go up in height, distances reduce.

Rough picture:



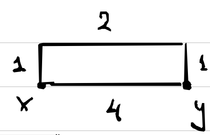
Definition: Let Γ be a graph. Define $\mathcal{H}(\Gamma)$ as follows.

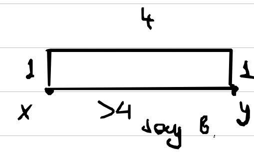




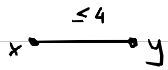
⇝ **Fact:** With respect to any graph Γ, $\mathcal{H}(\Gamma)$ is hyperbolic with $\delta = 10$.

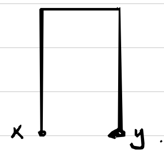
Geodesics: x, y same level.



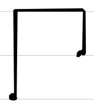


Generally:

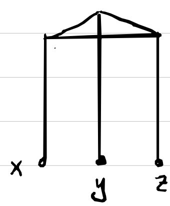




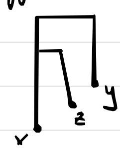
x, y are on different levels:



Geodesic triangles: x, y, z same level:



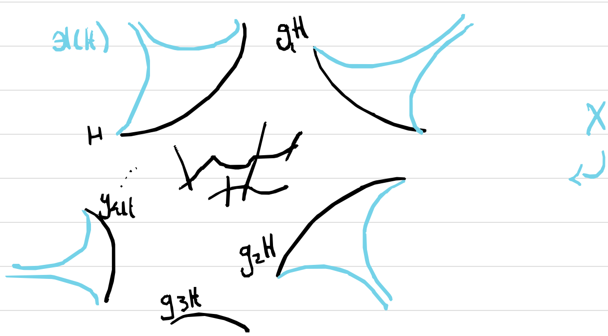
diff. levels:



Relatively hyperbolic groups:

⇝ Defn: Take $H \leq G$. Let $X = \mathrm{Cay}(G) \cup \bigcup_{g \in G/H} \mathcal{H}(gH)$, ie take all g-translates of H in $\mathrm{Cay}(G)$,
and glue horoballs to them.
We say G is hyperbolic relative to H if X is hyperbolic for some δ.

Cay G:

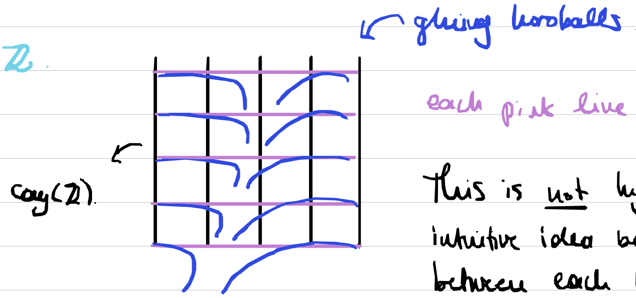


Examples:

1) $A * \mathbb{Z}$.

$A * \mathbb{Z}$ is hyperbolic rel. A $\forall A$.

2) $H \times \mathbb{Z}$.



each pink line is a copy of $Cay(H)$

This is not hyp. w.r.t. H

intuitive idea behind this is that there's gaps between each horoball.

---

2. Hyperbolically embedded subgroups:

⇝ Setup: $H \leq G$. Fix a subset $X \subseteq G$ st $G = \langle X \cup H \rangle$, the relative generating set of G wrt H.

Consider: $\Gamma = Cay(G, X \cup H)$.

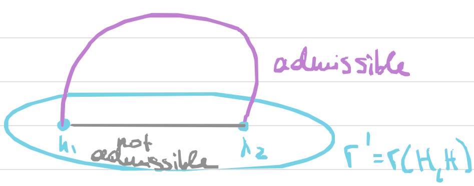
$\Gamma' = Cay(H, H) \subseteq \Gamma$ subgraph of Γ.

$|H|$ many vertices, edges connecting everything. (complete graph).

Defn: (Admissible): A path $p \in \Gamma$ is admissible if it avoids edges in Γ'.

Defn: (Relative metric): A relative metric is a path $\hat{d} : H \times H \to [0, \infty]$ such that $\hat{d}(h_1, h_2) = \min \{ \ell(p) \mid p \text{ is an admissible path in } \Gamma \text{ between } h_1 \text{ and } h_2 \}$.

length



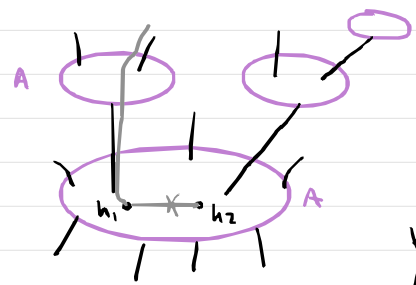
⇝ Main defn: H is hyperbolically embedded in G : $H \hookrightarrow_h (G, X)$ if
(i) $\Gamma(G, X \cup H)$ is hyperbolic
(ii) $(H, \hat{d})$ is proper, meaning every ball of finite radius is finite.

$H \hookrightarrow_h G$ if $H \hookrightarrow_h (G, X)$ for some X.

Before we mention the results that motivate the rest of this talk, we look at some examples:

⇝ EG: • $G \hookrightarrow_h G$ ⇝ we can take $\emptyset$ as generators.
• finite subgroups are always hyp. embedded gen. set. $X = G$ } degenerate cases.

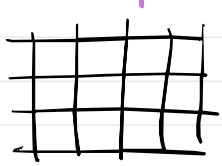
→ free product $A * \mathbb{Z}$ (A infinite)



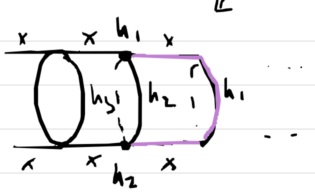
← no way to connect h_1, h_2 avoiding copies of A, because of the tree-like structure.

$\forall h_1 \neq h_2$, $\hat{d}(h_1, h_2) = \infty$.
All balls of finite radius have 1 element!

→ Non example: $H \times \mathbb{Z}$.



what happens when we take $X \cup H$
↓
$\langle x \rangle$.



QI to a line!

Question: what is the relationship between hyp. embedded subgroups and rel. hyp. groups?

Thm: H is hyp. embedded into G wrt a finite set $\Leftrightarrow$ G hyp. rel. H

Thm: G contains a non-degenerate subgroup, hyp embedded $\Leftrightarrow$ G acylindrically hyperbolic!

Acylindrical Hyperbolicity:

3) Acylindrical Actions: ⇝ Slightly weaker actions than proper + cocompact. introduced by Sela, defined by Bowditch.

<u>Setup:</u> Suppose we have:

$G \curvearrowright S$ ← not necessarily hyperbolic.

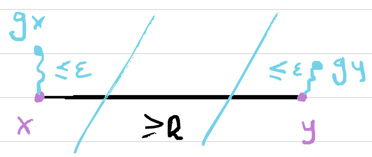
$\forall \varepsilon > 0$ and $\forall X \subseteq S$ we define the

$\varepsilon\text{-stab}_G(X) = \{g \in G \mid d(x, gx) \leq \varepsilon \ \ \forall x \in X\}$. (note $\varepsilon = 0$ gives $\text{stab}(X)$)

⇝ Defn: $G \curvearrowright S$ is proper $\iff \forall \varepsilon \ \forall s \in S$, $|\varepsilon\text{-stab}_G(X)| < \infty$

↳ equivalent definition to the one we saw in lectures!

Main Defn: $G \curvearrowright S$ is acylindrical if: $\forall \varepsilon > 0$, $\exists R, N \in \mathbb{N}$ such that $\forall x, y \in S$ that satisfy $d(x,y) \geq R$, we have that $|\varepsilon\text{-stab}_G(\{x,y\})| \leq N$.



← number of such elements is uniformly bounded by N.

Note that N does not depend on choices of x, y.

⇝ Intuitively, we can think of this as properness of S — a "thick diagonal" ie properness ignoring points that are close.

⇝ Examples/Remarks:

0) $G \curvearrowright pt$ acylindrical

1) $G \curvearrowright X$ is proper & cocompact then it is acylindrical.

2) Properness $\not\Rightarrow$ Acylindricity!

↳ what implies acylindricity above is cocompactness ⇒ gives uniformity of the bound N

3) $G \curvearrowright T$ tree. Edge stabilizers are finite, hence the action is cocomp. hence acylindrical

4. Acylindrically hyperbolic Groups:

⇝ **Defn:** A group G is acylindrically hyperbolic if G acts acylindrically and non-elementarily on a hyperbolic metric space.

↳ Why? ⇝ We don't have good descriptions of elementary actions.

What are elementary actions?
Elementary actions of G on a hyperbolic space are either:
1. Elliptic ⇝ have bounded orbits.
2. Parabolic
3. Lineal

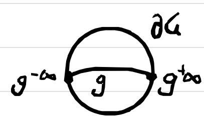
actions that do not produce "large" orbits
"simple" in a sense.

Q: Why is this a nice class of groups to study?
→ a lot of examples
→ can prove non-trivial results.

⇝ Examples:
1) Non-elementary hyp. groups = non virtually cyclic ⇒ non-elementary action.
2) Relatively hyperbolic groups.
3) $MCG(\Sigma_g)$ $g \geq 1$.
(We can also consider punctures with very few exeptions) (Bowditch)
4) $Out(F_n)$ $n \geq 2$
5) Groups of deficiency ≥ 2: This means ≥ 2 more generators than relators
6) π_1(compact 3-mflds) → either acyl. hyp
→ or virtually solvable
→ splits over an acyl. hyp. group.

Q: How do we find examples?

weak proper discontinuity
↓
Defn: $G \curvearrowright S$ $g \in G$. We say g satisfies the WPD condition if $\forall \varepsilon > 0$ $\exists M$ st $\forall N \geq M$
$$\{f \in G \mid d(s, fs) \leq \varepsilon, d(fg^N s, g^N s) \leq \varepsilon\} < \infty$$



→ If G acts on S (S hyperbolic) st $\exists$ at least one WPD loxodromic element then G is either virtually cyclic or acylindrically hyperbolic.

← finitely many of these!