Deformation Theory

Talk 8

Talk 8. Deformations of CM schemes of codimension 2.

Last time. We proved the following:

Thm. Let A be a regular local ring of $\dim A = n$, $B = A/I$, $\dim B = n-2$ and s.t. B is CM. There exists an exact sequence

$$0 \to A^{r} \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A \to B \to 0$$

where φ is given by a matrix of size $(r+1, r)$ s.t. its minors of rank r are a minimal set of generators of I.

Setting. Consider C, C' local Artin k-algebras and

$$0 \to J \to C' \to C \to 0$$

exact with $m_{C'} J = 0$. Take A' finite C'-algebra flat$/C'$, $A = A' \otimes_{C'} C$, $A_0 = A' \otimes_{C'} k$ and s.t. A_0 is smooth. Consider $B = A/I$ flat$/C$ and $B_0 = B \otimes_C k$. We will assume that B_0 has a resolution of the form

$$0 \to A_0^{r} \xrightarrow{\varphi_0} A_0^{r+1} \xrightarrow{f_0} A_0 \to B_0 \to 0$$

with f_0 given by the maximal minors of φ_0.

Theorem. (a) $\exists\ \varphi \in M_{(r+1)\times r}(A)$ with $r\times r$ minors f_i s.t. $I = \langle f_1, \dots, f_{r+1}\rangle$ and s.t. $\exists$ a resolution

$$0 \to A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A \to B \to 0$$

(b) If φ' is a lifting of φ to a matrix $\varphi' \in M_{(r+1)\times r}(A')$ and max. minors f_i', $\exists$ an exact sequence

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A' \to B' \to 0$$

with B' flat$/C'$ and $B' \otimes_{C'} C = B$.

(c) Any quotient $B' = A/I'$ flat$/C'$ and s.t. $B' \otimes_{C'} C = B$ arises as a lift of φ as in (b).

Proof. We prove (a) by induction on $\ell(C') = \mathrm{length}_C(C')$.
If $\ell(C') = 1 \rightsquigarrow C' = C$ and we take this as hypothesis (we assume resolution exists for k and then construct). Then the situation is that we have a tower of extensions

$$C^{n+1} \longrightarrow C^n \longrightarrow \cdots \longrightarrow C_1 \to C$$

s.t. $\ell(C^k) = k$ and we assume (a) is true for C^n and consider B^{n+1} a lifting of B^n. By (c) this lifting is given by a resolution as in (b).

Now we prove (b) assuming (a). Take φ' any lifting of φ, and consider the complex

$$L'_\bullet : A'^{r} \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A'$$

(it is a complex by the same argument as in the structure thm).

Since A' is flat over C' we have an exact sequence of complexes

$$0 \to L'_\bullet \otimes_{C'} J \to L'_\bullet \to L'_\bullet \otimes_{C'} C \to 0 \quad (*)$$

Notice that $L'_\bullet \otimes_{C'} C$ is just

$$L_\bullet : A^{r} \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A, \quad \text{since}$$

$$A'^{r} \otimes_{C'} C = (A' \otimes_{C'} C)^{r} = A^{r} \quad \text{and}$$

$$L'_\bullet \otimes_{C'} J : A'^{r} \otimes_{C'} J \to A'^{r+1} \otimes_{C'} J \to A' \otimes_{C'} J$$

$$A' \otimes_{C'} J = A' \otimes_{C'} (C \otimes_C J) = (A' \otimes_{C'} C) \otimes_C J = A \otimes_C J$$

$$\rightsquigarrow L'_\bullet \otimes_{C'} J = L_\bullet \otimes_C J.$$

Part (a) gives exactness of $L_\bullet$, and cokernel of $L_\bullet$ is B. $L_\bullet \otimes_C J$ is also exact with cokernel $B \otimes_C J$ because of the following lemma.

Exercise. Let $M_\bullet : 0 \to M_1 \to M_2 \to \dots$ be an exact sequence of flat modules. Then, for any mod. N, $M_\bullet \otimes N$ is exact.

This implies $L'_\bullet$ is also exact. We denote its cokernel by B' so we have

$$0 \to A'^{r} \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A' \to B' \to 0$$

Considering the homology l.e.s of (*)

$$H_1(L_\bullet) \to H_0(L_\bullet \otimes J) \to H_0(L'_\bullet) \to H_0(L_\bullet) \to 0$$

and $H_0(L_\bullet) = A/\operatorname{Im} f = B$

$H_0(L'_\bullet) = B'$

$H_0(L_\bullet \otimes J) = B \otimes J$

$H_1(L_\bullet) = 0$

so we obtain the exact sequence

$$0 \to B \otimes_C J \to B' \to B \to 0,$$

and injectivity of $B \otimes_C J \to B'$ implies B' is flat over C'.

(c) Assume $B' = A'/I'$ is flat $/C'$ and $B' \otimes_{C'} C = B$. We can consider $g_i \in I'$ a lift of each $f_i \in B$, and notice by Nakayama's lemma $I' = \langle g_1, \dots, g_{r+1} \rangle$, as we have

$$I'/JI' = I' \otimes_{C'} C = I$$

Then we obtain a resolution

$$0 \to M \to A'^{r+1} \xrightarrow{g} A' \to B' \to 0$$

Since B' is flat$/C'$, so is M. Then

$$0 \to M \otimes_{C'} C \to A^{r+1} \to A \to B \to 0$$

is exact $\rightsquigarrow$ $M \otimes_{C'} C \simeq A^r$. Take an A-basis of $M \otimes_{C'} C$ and lift to M, so by Nakayama's lemma we obtain an exact seq.

$$0 \to K \to A'^r \to M \to 0$$

and tensoring $\otimes_{C'} C$ we see $K \otimes_{C'} C = 0$, and by Nakayama's lemma $K = 0 \rightsquigarrow M \simeq A'^r$.

We obtained a resolution

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{g} A \to B \to 0$$

with φ' a lifting of φ, so by (b) $\exists$

$$0 \to A'^r \xrightarrow{\varphi'} A'^{r+1} \xrightarrow{f'} A' \to B'' \to 0$$

for another lifting B'' of B. We need to prove that $B' = B''$ to conclude.

Lemma. Let A be a C-alg. flat$/C$, $A \otimes_C k$ normal. If $Z \subseteq X = \operatorname{Spec} A$ has $\operatorname{codim} Z \geq 2$ then $H^0(X \setminus Z, \mathcal{O}_X) = A$.

As in the structure theorem, $I' \simeq I''$ as A'-mods.
For $X = \operatorname{Spec} A'$, $Z = \operatorname{Supp} B$ and $Z \hookrightarrow X$ (under the inclusion $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$), and notice that $Z = \operatorname{Supp}(B') = \operatorname{Supp}(B'')$ since

$$B \otimes_C k = B' \otimes_C k = B'' \otimes_C k = B_0 .$$

Consider the isomorphism of sheaves $\varphi : \mathcal{I}' \to \mathcal{I}''$ and notice that $\mathcal{I}'|_{X \setminus Z} \simeq \mathcal{I}''|_{X \setminus Z} \simeq \mathcal{O}_{X \setminus Z}$, so in $X \setminus Z$ φ corresponds to $f \in \Gamma(X \setminus Z, \mathcal{O}_X) = A'$ and by same argument as in the theorem of CM rings, $f \in A'^{\times}$, so $I' = f I''$ differs by a unit $\rightsquigarrow$ $B' = B''$. ∎

Corollary. If $Y_0 \subseteq X_0$ is a CM closed subsch. of codim 2 and $Y \subseteq X$ is a deformation of Y_0, then the obstruction to the existence of an extension Y' lives in $H^1(Y_0, N_{Y_0/X_0} \otimes J)$.

Remark. This was already proved in Talk 5, under the hypothesis of existence of local defs. (that is true in this case).

Remark (complete intersections). Recall that a quotient $B = A/I$ is a complete intersection if $I = \langle a_1, \dots, a_r \rangle$ with $r = \dim A - \dim B$. For A a CM ring, we can construct the Koszul complex

$$K_\bullet(a_1, \dots, a_r) : 0 \to \Lambda^r A^r \to \Lambda^{r-1} A \to \dots \to A^r \to A \to B \to 0$$

that in this case gives a resolution on B.
Same proof of the previous theorem (but simpler) shows that deformations of B arise as liftings of the Koszul complex, so we obtain that defs. of l.c.i are locally unobstructed in any codimension. Moreover, defs. of l.c.i are themselves l.c.i.

Important. The fact that defs. are locally unobstructed is not true for CM schemes of codim > 2.